k-EXTREME POINTS IN SYMMETRIC SPACES OF MEASURABLE OPERATORS

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Abstract. Let $\mathcal{M}$ be a semifinite von Neumann algebra with a faithful, normal, semifinite trace τ and E be a strongly symmetric Banach function space on $[0, \tau(\mathbf{1}))$. We show that an operator x in the unit sphere of $E(\mathcal{M}, \tau)$ is k-extreme, $k \in \mathbb{N}$, whenever its singular value function $\mu(x)$ is k-extreme and one of the following conditions hold (i) $\mu(\infty, x) = \lim_{t\to\infty} \mu(t, x) = 0$ or (ii) $n(x)\mathcal{M}n(x^*) = 0$ and $|x| \geq \mu(\infty, x)s(x)$, where $n(x)$ and $s(x)$ are null and support projections of x, respectively. The converse is true whenever $\mathcal{M}$ is non-atomic. The global k-rotundity property follows, that is if $\mathcal{M}$ is non-atomic then E is k-rotund if and only if $E(\mathcal{M}, \tau)$ is k-rotund. As a consequence of the noncommutive results we obtain that f is a k-extreme point of the unit ball of the strongly symmetric function space E if and only if its decreasing rearrangement $\mu(f)$ is k-extreme and $|f| \geq \mu(\infty, f)$. We conclude with the corollary on orbits $\Omega(g)$ and $\Omega'(g)$. We get that f is a k-extreme point of the orbit $\Omega(g)$, $g \in L_1 + L_\infty$, or $\Omega'(g)$, $g \in L_1[0, \alpha)$, $\alpha < \infty$, if and only if $\mu(f) = \mu(g)$ and $|f| \geq \mu(\infty, f)$. From this we obtain a characterization of k-extreme points in Marcinkiewicz spaces.

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1. Preliminaries

The purpose of the article is to characterize k-extreme elements of the symmetric spaces of measurable operators in terms of their singular value functions. There have been many results in the noncommutative spaces $E(\mathcal{M}, \tau)$, in particular in the noncommutative L_p spaces, aiming to reduce the study on their geometric properties to the commutative settings. The first research in this direction can be attributed to Arazy [1], who described extreme points in the unitary matrix spaces. In the symmetric spaces of measurable operators extreme points were characterized by Chillin, Krygin and Sukochev in [6].

The theory of noncommutative spaces $E(\mathcal{M}, \tau)$ has attracted many mathematicians who continue to research their geometric properties. There is a long list of papers relating geometric properties of the operators and their decreasing rearrangements, see for example results on local uniform rotundity and uniform rotundity [7], complex extreme points [9], Kadets-Klee property [5], or smooth points in [10]. Worth mentioning is also work in [20] investigating local geometrical aspects

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of subspaces of noncommutative L_p spaces associated with general von Neumann algebras.

Let $\mathcal{M}$ be a semifinite von Neumann algebra on the Hilbert space H with given faithful normal semifinite trace τ. The symbol $\mathbf{1}$ will be used to denote the identity in $\mathcal{M}$ and $\mathcal{P}(\mathcal{M})$ will stand for the set of all projections in $\mathcal{M}$. It is known that $\mathcal{P}(\mathcal{M})$ is a complete lattice, that is the supremum and infimum exist for any non-empty subset of $\mathcal{P}(\mathcal{M})$. Given $p, q \in \mathcal{P}(\mathcal{M})$, we will write $p \vee q$ and $p \wedge q$ to denote the supremum and infimum of p and q, respectively. The projections p and q are said to be equivalent (relative to the von Neumann algebra $\mathcal{M}$) denoted by $p \sim q$, if there exists a partial isometry $v \in M$ such that $p = v^*v$ and $q = vv^*$. A non-zero projection $p \in \mathcal{P}(\mathcal{M})$ is called *minimal* if $q \in \mathcal{P}(\mathcal{M})$ and $q \leq p$ imply that $q = p$ or $q = 0$. The von Neumann algebra $\mathcal{M}$ is called *non-atomic* if it has no minimal projections. A projection $p \in P(\mathcal{M})$ is called *σ-finite* (with respect to the trace τ) if there exists a sequence $\{p_n\}$ in $P(\mathcal{M})$ such that $p_n \uparrow p$ and $\tau(p_n) < \infty$. If the unit element $\mathbf{1}$ in $\mathcal{M}$ is σ-finite, then we say that the trace τ on $\mathcal{M}$ is σ-finite [15, 25, 13].

Given a self-adjoint (possibly unbounded) linear operator $x : \mathcal{D}(x) \subset H \to H$, we denote by $e^x(\cdot)$ the spectral measure of x. A closed, densely defined operator x, which commutes with all the unitary operators from the commutant $\mathcal{M}'$ of $\mathcal{M}$, is said to be *affiliated* with $\mathcal{M}$. If in addition there exists $\lambda > 0$ such that $\tau\left(e^{|x|}(\lambda, \infty)\right) < \infty$ then x is called *τ-measurable.* The collection of all τ-measurable operators is denoted by $S\left(\mathcal{M}, \tau\right)$.

For an operator $x \in S\left(\mathcal{M}, \tau\right)$, the function $\mu(x) = \mu(\cdot, x) : [0, \infty) \to [0, \infty]$ defined by

$$\mu(t, x) = \inf\left\{s \geq 0 : \tau(e^{|x|}(s, \infty)) \leq t\right\}, \quad t > 0,$$

is called a *decreasing rearrangement* of x or a *generalized singular value function* of x. It follows that $\mu(x)$ is decreasing, right-continuous function on the real half-line. Note that in this paper the terms decreasing or increasing will always mean non-increasing or non-decreasing, respectively. Observe that if $\tau(\mathbf{1}) < \infty$ then $\mu(t, x) = 0$ for all $t \geq \tau(\mathbf{1})$, and so $\mu(\infty, x) = \lim_{t\to\infty} \mu(t, x) = 0$. We denote by $S_0\left(\mathcal{M}, \tau\right)$ the collection of all $x \in S\left(\mathcal{M}, \tau\right)$ for which $\mu(\infty, x) = 0$. $S_0\left(\mathcal{M}, \tau\right)$ is a $*$-subalgebra in $S\left(\mathcal{M}, \tau\right)$.

By a *positive operator* x we mean a self-adjoint operator such that $\langle x\xi, \xi\rangle \geqslant 0$ for all ξ in the domain of x. $S^+\left(\mathcal{M}, \tau\right)$ will stand for the cone of positive measurable operators.

Any closed, densely defined linear operator x has the *polar decomposition* $x = u\,|x|$, where u is a partial isometry with kernel $\operatorname{Ker} u = \operatorname{Ker} x$. Moreover, the polar decomposition of x^* is given by $x^* = u^*\,|x^*|$. We have that $u^*u = s(x) = e^{|x|}(0, \infty)$ and $uu^* = s(x^*) = e^{|x^*|}(0, \infty)$, where *support projections* $s(x)$ and $s(x^*)$ are projections onto $\operatorname{Ker}^{\perp} x$ and $\operatorname{Ker}^{\perp} x^*$, respectively. It is known that x is τ-measurable if and only if $u \in \mathcal{M}$ and $|x|$ is τ-measurable. The *null projection* $n(x) = \mathbf{1} - s(x) = e^{|x|}\{0\}$ is a projection onto $\operatorname{Ker} x$.

Let $L^0 = L^0[0, \alpha)$ stand for the space of all complex-valued Lebesgue measurable functions on $[0, \alpha)$, $\alpha \leq \infty$, (with identification a.e. with respect to the Lebesgue measure m).

Given $f \in L^0$, the *distribution function* $d(f)$ of f is defined by $d(\lambda, f) = m\{t > 0 : |f(t)| > \lambda\}$, for all $\lambda \geq 0$. The *decreasing rearrangement* of f is given by $\mu(t, f) = \inf\{s > 0 : d(s, f) \leq t\}$, $t \geq 0$. Observe that $d(f) = d(\cdot, f)$ and

$\mu(f) = \mu(\cdot, f)$ are right-continuous, decreasing functions on $[0, \infty)$. A *support* of function $f \in E$, that is the set $\{t \in [0, \alpha) : f(t) \neq 0\}$ will be denoted by $\operatorname{supp} f$. Moreover for $f, g \in L^0$, we will write $f \prec g$ if $\int_0^t \mu(f) \leq \int_0^t \mu(g)$ for all $t > 0$. For operators $x, y \in S(\mathcal{M}, \tau)$, $x \prec y$ denotes $\mu(x) \prec \mu(y)$.

A Banach space $E \subset L^0$ is called *symmetric* (also called *rearrangement invariant*) if it follows from $f \in L^0$, $g \in E$ and $\mu(f) \leq \mu(g)$ that $f \in E$ and $\|f\|_E \leq \|g\|_E$ [3, 18]. It is well known that for every symmetric space E on $[0, \alpha)$, we have that $E \subset L_1[0, \alpha) + L_\infty[0, \alpha)$. Moreover, if from $f, g \in E$ and $f \prec g$ we have that $\|f\|_E \leq \|g\|_E$ then E is called a *strongly symmetric* function space. Any symmetric space which is order continuous or satisfies the Fatou property is strongly symmetric [3, 18]. Throughout the remainder of the paper we will assume that E is strongly symmetric function space.

Given a semifinite von Neumann algebra $(\mathcal{M}, \tau)$ and a symmetric Banach function space E on $[0, \alpha)$, $\alpha = \tau(\mathbf{1})$, the corresponding *noncommutative space* $E(\mathcal{M}, \tau)$ is defined by setting

$$E(\mathcal{M}, \tau) = \{x \in S(\mathcal{M}, \tau) : \quad \mu(x) \in E\}.$$

Observe that for any $x \in S(\mathcal{M}, \tau)$, $\mu(x) = \mu(x)\chi_{[0,\alpha)}$, so we identify those two functions. $E(\mathcal{M}, \tau)$ equipped with the norm $\|x\|_{E(M,\tau)} = \|\mu(x)\|_E$ is a Banach space [16] and it is called *symmetric space of measurable operators* associated with $(\mathcal{M}, \tau)$ and corresponding to E.

The trace τ on $\mathcal{M}^+$ extends uniquely to an additive, positively homogeneous, unitarily invariant and normal functional $\tilde{\tau} : S(\mathcal{M}, \tau)^+ \to [0, \infty]$, which is given by $\tilde{\tau}(x) = \int_0^\infty \mu(x)$, $x \in S(\mathcal{M}, \tau)^+$. This extension is also denoted by τ. The simple observation that an operator $x \in E(\mathcal{M}, \tau)$ is trace commuting with any projections $p \in P(\mathcal{M})$, that is $\tau(xp) = \tau(px)$, will be used further in the paper [12].

If $E = L_p$ on $[0, \tau(\mathbf{1}))$, $1 \leqslant p < \infty$, then for $x \in L_p(\mathcal{M}, \tau)$ we have $\|x\|_{L_p(\mathcal{M},\tau)} = \|\mu(x)\|_{L_p} = \left(\int_0^{\tau(\mathbf{1})} \mu(|x|^p)\right)^{1/p} = (\tau(|x|^p))^{1/p}$. The space $L_1(\mathcal{M}, \tau) + \mathcal{M}$ is the space of all operators $x \in S(\mathcal{M}, \tau)$ for which $\int_0^1 \mu(x) < \infty$. By [12, Proposition 2.6] we have that $E(\mathcal{M}, \tau) \subset L_1(\mathcal{M}, \tau) + \mathcal{M}$.

Consider a commutative von Neumann algebra $\mathcal{N} = \{N_f : L_2[0, \tau(\mathbf{1})) \to L_2[0, \tau(\mathbf{1})) : f \in L_\infty[0, \tau(\mathbf{1}))\}$, where N_f acts via pointwise multiplication on $L_2[0, \tau(\mathbf{1}))$ and the trace η is given by integration, that is $N_f(g) = f \cdot g$, $g \in L_2[0, \tau(\mathbf{1}))$, and $\eta(N_f) = \int_0^{\tau(\mathbf{1})} f dm$, where m is the Lebesgue measure on $[0, \tau(\mathbf{1}))$. This von Neumann algebra is commonly identified with $L_\infty[0, \tau(\mathbf{1}))$. $S(\mathcal{N}, \eta)$ is identified with the set of all measurable functions on R^+ which are bounded except on a set of finite measure, denoted in this paper by $S([0, \tau(\mathbf{1})), m)$. Moreover for $N_f \in S(\mathcal{N}, \eta)$, the generalized singular value function $\mu(N_f)$ is precisely the decreasing rearrangement $\mu(f)$ of the function $f \in S([0, \tau(\mathbf{1})), m)$. For any symmetric function space E we have that $E(\mathcal{N}, \eta)$ is isometrically isomorphic to the function space E.

For the theory of operator algebras we refer to [15, 25], and for noncommutative Banach function spaces to [11, 13, 19].

For the readers' convenience we will recall below the well known result on commuting elements of $S(\mathcal{M}, \tau)$.

Proposition 1.1. [13] *Two self-adjoint elements $a, b \in S(\mathcal{M}, \tau)$ commute if and only if $e^a(s, \infty)e^b(s, \infty) = e^b(s, \infty)e^a(s, \infty)$ for all $s > 0$.*

The following facts will be used in the subsequent sections.

Lemma 1.2. [11, Lemma 2.5], [13] *Suppose that the von Neumann algebra $\mathcal{M}$ is non-atomic and $a \in S^+(\mathcal{M},\tau)$. If $\lambda < \tau(\mathbf{1})$ then there exists $e \in P(\mathcal{M})$ such that*

$$e^a\left(\mu(\lambda,a),\infty)\right) \le e \le e^a\left[\mu(\lambda,a),\infty)\right) \quad \text{and} \quad \tau(e)=\lambda.$$

Lemma 1.3. [11, Lemma 3.2], [13] *Suppose that $a \in S^+(\mathcal{M},\tau)$, $e \in P(\mathcal{M})$ is such that $\tau(e) < \infty$ and*

$$e^a\left(\lambda,\infty\right) \le e \le e^a[\lambda,\infty).$$

Then $ae = ea = eae$ and $\mu(ae) = \mu(a)\chi_{[0,\tau(e))}$. In particular,

$$\tau(eae) = \tau(ae) = \int_0^{\tau(e)} \mu(a).$$

Lemma 1.4. [13] *Suppose that $0 \le x \in L_1(\mathcal{M},\tau) + \mathcal{M}$. Let $\lambda > 0$ and e be a projection in $\mathcal{M}$ such that $\tau(e) = \lambda$ and*

$$\tau(xe) = \int_0^{\lambda} \mu(x).$$

Then

$$e^x\left(\mu(\lambda,x),\infty\right) \le e \le e^x\left[\mu(\lambda,x),\infty\right).$$

Proposition 1.5.
(1) [9, Proposition 1.1] *For $x \in S(\mathcal{M},\tau)$,*

$$\mu(|x| + \mu(\infty,x)n(x)) = \mu(x)$$

.
(2) [9, Proposition 1.1] *If $x \in S(\mathcal{M},\tau)$ and $|x| \geqslant \mu(\infty,x)s(x)$ then $\mu(|x| - \mu(\infty,x)s(x)) = \mu(x) - \mu(\infty,x)$.*
(3) [13], [10, Lemma 1.3] *If $x,y \in L_1(\mathcal{M},\tau) + \mathcal{M}$, $\mu(\infty,x) = \mu(\infty,y) = 0$ and $\mu((x+y)/2) = \mu(x) = \mu(y)$, then $x = y$.*
(4) [22] *If $x,y \in S(\mathcal{M},\tau)$, $x = x^*$, $y \ge 0$ and $-y \le x \le y$, then $x \prec y$.*
(5) [6, Proposition 2.2] *If $x,y \in S^+(\mathcal{M},\tau)$, $y \ne 0$ and $x \ge \mu(\infty,x)\mathbf{1}$, then there exists $s > 0$ such that $\mu(s,x) < \mu(s,x+y)$.*
(6) [6, Proposition 3.5] *If $x,y \in S(\mathcal{M},\tau)$, $y = y^*$, $x \ge \mu(\infty,x)\mathbf{1}$ and $\mu(x+iy) = \mu(x)$, then $y = 0$.*
(7) [13] *If $s \ge 0$ and $p = e^{|x|}(s,\infty)$ then $\mu(|x|\,p) = \mu(x)\chi_{[0,\tau(p))}$.*

Condition (1) in the above proposition implies the following.

Corollary 1.6. *Let $x \in S(\mathcal{M},\tau)$ and $p \in \mathcal{P}(\mathcal{M})$. If $px = xp = 0$ and $0 \le C \le \mu(\infty,x)$ then $\mu(x+Cp) = \mu(x)$.*

Proof. Suppose that $px = xp = 0$ and $0 \le C \le \mu(\infty,x)$. Then $n(x) \ge p$, $n(x^*) \ge p$ and $|x + Cp| = |x| + Cp$. The claim follows now by Proposition 1.5 (1). Indeed, $\mu(|x|) \le \mu(|x| + Cp) = \mu(x+Cp) \le \mu(|x| + \mu(\infty,x)n(x)) = \mu(x)$. □

Below we include the results that show that E is isometrically embedded into $E(\mathcal{M},\tau)$, if certain conditions on the trace τ and von Neumann algebra $\mathcal{M}$ are imposed.

Recall that given two $*$-algebras $\mathcal{A}$ and $\mathcal{B}$, the mapping $\Phi : \mathcal{A} \to \mathcal{B}$ is called a *$*$-homomorphism* if Φ is an algebra homomorphism and $\Phi(x^*) = \Phi(x)^*$ for all $x \in \mathcal{A}$. If, in addition, $\mathcal{A}$ and $\mathcal{B}$ are unital and $\Phi(\mathbf{1}_A) = \mathbf{1}_B$ then Φ is called *unital $*$-homomorphism*.

Proposition 1.7. [13], [7, Lemma 1.3] *Let $\mathcal{M}$ be a non-atomic von Neumann algebra with a faithful, normal, σ-finite trace τ and $x \in S_0^+(\mathcal{M},\tau)$. Then there exists a non-atomic commutative von Neumann subalgebra $\mathcal{N}$ in $\mathcal{M}$ and a $*$-isomorphism U from the $*$-algebra $S(\mathcal{N},\tau)$ onto the $*$-algebra $S([0,\tau(\mathbf{1})),m)$ such that $x \in S(\mathcal{N},\tau)$ and $\mu(y) = \mu(Uy)$ for every $y \in S(\mathcal{N},\tau)$.*

Given an operator $x \in S(\mathcal{M},\tau)$ and a projection $p \in \mathcal{P}(\mathcal{M})$ we define the von Neumann algebra $\mathcal{M}_p = \{py|_{p(H)} : \quad y \in \mathcal{M}\}$. It is known that there is a unital $*$-isomorphism from $S(\mathcal{M}_p,\tau_p)$ onto $pS(\mathcal{M},\tau)\,p$. Moreover, the decreasing rearrangement μ^{τ_p} computed with respect to the von Neumann algebra $(\mathcal{M}_p,\tau_p)$ is given by $\mu^{\tau_p}(y) = \mu(pyp)$, $y \in S(\mathcal{M}_p,\tau_p)$. See [9, 10, 11] for details.

Using the theory of measure preserving transformations which retrieve functions from their decreasing rearrangements and U^{-1} from Proposition 1.7 the following can be shown.

Proposition 1.8. [13] *Suppose that $\mathcal{M}$ is a non-atomic von Neumann algebra with a faithful, normal trace τ. Let $x \in (L_1(\mathcal{M},\tau)+\mathcal{M}) \cap S_0^+(\mathcal{M},\tau)$. Then there exist a non-atomic commutative von Neumann subalgebra $\mathcal{N} \subset s(x)\mathcal{M}s(x)$ and a unital $*$-isomorphism V acting from the $*$-algebra $S([0,\tau(s(x))),m)$ into the $*$-algebra $S(\mathcal{N},\tau)$, such that*

$$V\mu(x) = x \quad \textit{and} \quad \mu(V(f)) = \mu(f) \quad \textit{for all } f \in S([0,\tau(s(x))),m)\,.$$

Proposition 1.9. [13] *Suppose that $\mathcal{M}$ is a non-atomic von Neumann algebra with a faithful, normal, σ-finite trace τ. Let $x \in (L_1(\mathcal{M},\tau)+\mathcal{M}) \cap S_0^+(\mathcal{M},\tau)$ and $\tau(s(x)) < \infty$. Then there exist a non-atomic commutative von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$ and a unital $*$-isomorphism V acting from the $*$-algebra $S([0,\tau(\mathbf{1})),m)$ into the $*$-algebra $S(\mathcal{N},\tau)$, such that*

$$V\mu(x) = x \quad \textit{and} \quad \mu(V(f)) = \mu(f) \quad \textit{for all } f \in S([0,\tau(\mathbf{1})),m)\,.$$

We will need further a specific version of the above propositions.

Corollary 1.10. *Let $\mathcal{M}$ be a non-atomic von Neumann algebra with a faithful, normal, σ-finite trace τ, $x \in S(\mathcal{M},\tau)$, and $|x| \geq \mu(\infty,x)s(x)$. Denote by $p = s(|x| - \mu(\infty,x)s(x))$ and define projection $q \in \mathcal{P}(\mathcal{M})$ in the following way.*

(i) *If $\tau(s(x)) < \infty$ set $q = \mathbf{1}$.*
(ii) *If $\tau(s(x)) = \infty$ and $\tau(p) < \infty$, set $q = s(x)$.*
(iii) *If $\tau(p) = \infty$, set $q = p$.*

Then there exist a non-atomic commutative von Neumann subalgebra $\mathcal{N} \subset q\mathcal{M}q$ and a unital $*$-isomorphism V acting from the $*$-algebra $S([0,\tau(\mathbf{1})),m)$ into the $*$-algebra $S(\mathcal{N},\tau)$, such that

$$V\mu(x) = |x|\,q \quad \textit{and} \quad \mu(V(f)) = \mu(f).$$

for all $f \in S([0,\tau(\mathbf{1})),m)$.

Proof. Observe that $p = s(|x| - \mu(\infty,x)s(x)) = e^{|x|}(\mu(\infty,x),\infty) \leq s(x)$. Hence if $\tau(p) = \infty$ then also $\tau(s(x)) = \infty$, and therefore conditions (i), (ii), and (iii) give all possible cases. Furthermore, by Proposition 1.5 (2), $\mu(|x| - \mu(\infty,x)s(x)) = \mu(x) - \mu(\infty,x)$, and so $|x| - \mu(\infty,x)s(x) \in S_0^+(\mathcal{M},\tau)$.

Note that in either case $\tau(q) = \tau(\mathbf{1})$. Hence in view of Proposition 1.5 (7), it follows that $\mu(|x|\,q) = \mu(x)\chi_{[0,\tau(q))} = \mu(x)$.

Case (i). Since $\tau(s(x)) < \infty$, we have that $\mu(\infty, x) = 0$. Therefore the claim is an immediate consequence of the Proposition 1.9 applied to $|x|$.

Case (ii). Let $\tau(s(x)) = \infty$, $\tau(p) < \infty$ and $q = s(x)$. Applying Proposition 1.9 to the operator $|x| - \mu(\infty, x)s(x) = s(x)(|x| - \mu(\infty, x)s(x))s(x) \in s(x)S(\mathcal{M}, \tau)\, s(x)$ and to the von Neumann algebra $s(x)\mathcal{M}s(x)$, there exists a non-atomic commutative von Neumann algebra $\mathcal{N} \subset s(x)\mathcal{M}s(x)$ and a $*$-isomorphism V from $S([0, \tau(s(x))), m) = S([0, \infty), m)$ into $S(\mathcal{N}, \tau)$ such that

$$V\mu(|x| - \mu(\infty, x)s(x)) = |x| - \mu(\infty, x)s(x) \text{ and } \mu(V(f)) = \mu(f)$$

for all $f \in S([0, \infty), m)$. Since $V(\chi_{[0,\infty)}) = s(x)$,

$$\begin{aligned} |x| - \mu(\infty, x)s(x) &= V\mu(|x| - \mu(\infty, x)s(x)) = V(\mu(x) - \mu(\infty, x)) \\ &= V\mu(x) - \mu(\infty, x)V(\chi_{[0,\infty)}) = V\mu(x) - \mu(\infty, x)s(x), \end{aligned}$$

and consequently $V\mu(x) = |x| = |x|\, s(x)$.

Case (iii). Assume that $\tau(p) = \infty$ and $q = p$. By Proposition 1.8 applied to the operator $|x| - \mu(\infty, x)s(x)$ and von Neumann algebra $\mathcal{M}$, there exists a non-atomic commutative von Neumann algebra $\mathcal{N} \subset p\mathcal{M}p$ and a $*$-isomorphism V from $S([0, \tau(p)), m) = S([0, \infty), m)$ into $S(\mathcal{N}, \tau)$ such that

$$V\mu(|x| - \mu(\infty, x)s(x)) = |x| - \mu(\infty, x)s(x) \text{ and } \mu(V(f)) = \mu(f)$$

for all $f \in S([0, \infty), m)$. Since $p \leq s(x)$,

$$|x| - \mu(\infty, x)s(x) = (|x| - \mu(\infty, x)s(x))p = |x|\, p - \mu(\infty, x)p$$

and $V(\chi_{[0,\infty)}) = p$. Thus again we have

$$\begin{aligned} |x|\, p - \mu(\infty, x)p = |x| - \mu(\infty, x)s(x) &= V\mu(|x| - \mu(\infty, x)s(x)) \\ &= V(\mu(x) - \mu(\infty, x)) = V\mu(x) - \mu(\infty, x)V(\chi_{[0,\infty)}) \\ &= V\mu(x) - \mu(\infty, x)p, \end{aligned}$$

and $V\mu(x) = |x|\, p$. □

Remark 1.11. We will describe below the construction of a non-atomic von Neumann algebra $\mathcal{A}$ with the trace κ, such that $E(\mathcal{M}, \tau)$ embeds isometrically into $E(\mathcal{A}, \kappa)$, for any symmetric function space E.

Let $\mathcal{A} = \mathcal{N}\overline{\otimes}\mathcal{M}$ be a tensor product of von Neumann algebras $\mathcal{N}$ and $\mathcal{M}$, where $\mathcal{N}$ is a commutative von Neumann algebra identified with $L_\infty[0, 1]$ with the trace η. Let $\kappa = \eta \otimes \tau$ be a tensor product of the traces η and τ, that is $\kappa(N_f \otimes x) = \eta(N_f)\tau(x)$, [15, 25]. It is well known that $(\mathcal{A}, \kappa)$ has no atoms.

Let $\mathbb{1}$ be the identity operator on $L^2[0, 1]$ and denote by $\mathbb{C}\mathbb{1} = \{\lambda\mathbb{1} : \lambda \in \mathbb{C}\}$. Let $x \in S(\mathcal{M}, \tau)$ and consider a linear subspace D in $L_2[0, 1] \otimes H$ generated by the vectors of the form $\zeta \otimes \xi$, where $\zeta \in L_2[0, 1]$ and $\xi \in D(x) \subset H$. For every $\alpha = \sum_{i=1}^{n} \zeta_i \otimes \xi_i \in D$ define $(\mathbb{1} \otimes x)(\alpha) = \sum_{i=1}^{n} \zeta_i \otimes x\xi_i$. The linear operator $\mathbb{1} \otimes x : D \to L_2[0, 1] \otimes H$ with domain D is preclosed, and by Lemma 1.2 in [7] its closure $\mathbb{1}\overline{\otimes}x$ is contained in $S(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa)$.

The map $\pi : x \to \mathbb{1} \otimes x$, $x \in \mathcal{M}$, is a unital trace preserving $*$-isomorphism from $\mathcal{M}$ onto the von Neumann subalgebra $\mathbb{C}\mathbb{1} \otimes \mathcal{M}$. Consequently, π extends uniquely to a $*$-isomorphism $\tilde{\pi}$ from $S(\mathcal{M}, \tau)$ onto $S(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa)$ [13]. In fact one can show that $\tilde{\pi}(x) = \mathbb{1}\overline{\otimes}x$.

Since every $*$-homomorphism is an order preserving map, $x \geq 0$ if and only if $\mathbb{1}\overline{\otimes}x \geq 0$, where $x \in S(\mathcal{M}, \tau)$. The spectral measure $e^{\tilde{\pi}(x)}$ of $\tilde{\pi}(x)$ is given by $e^{\tilde{\pi}(x)}(B) = \pi(e^x(B))$, that is $e^{\mathbb{1}\overline{\otimes}x}(B) = \mathbb{1} \otimes e^x(B)$ for any Borel set $B \subset \mathbb{R}$. Hence $\kappa\left(e^{\mathbb{1}\overline{\otimes}x}(s,\infty)\right) = \kappa\left(\mathbb{1} \otimes e^x(s,\infty)\right) = \tau(e^x(s,\infty))$ for any $s > 0$. Consequently $\tilde{\pi}$ preserves the singular value function in the sense that $\tilde{\mu}(\mathbb{1}\overline{\otimes}x) = \mu(x)$, where $\tilde{\mu}(\mathbb{1}\overline{\otimes}x)$ is the singular value function of $\mathbb{1}\overline{\otimes}x$ computed with respect to the von Neumann algebra $\mathbb{C}\mathbb{1} \otimes \mathcal{M}$ and the trace κ.

Thus

$$\|\tilde{\pi}(x)\|_{E(\mathbb{C}\mathbb{1}\otimes\mathcal{M},\kappa)} = \|\tilde{\mu}(\mathbb{1}\overline{\otimes}x)\|_E = \|\mu(x)\|_E = \|x\|_{E(M,\tau)},$$

where

$$\begin{aligned} E(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa) &= \{\mathbb{1}\overline{\otimes}x \in S(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa) : \tilde{\mu}(\mathbb{1}\overline{\otimes}x) \in E\} \\ &= \{\mathbb{1}\overline{\otimes}x : x \in S(\mathcal{M}, \tau) \text{ and } \mu(x) \in E\}. \end{aligned}$$

Hence $\tilde{\pi}$ is a $*$-isomorphism which is also an isometry from $E(\mathcal{M}, \tau)$ onto $E(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa)$. We refer reader to [7, 13, 24] for details.

Given a natural number k, consider a normed linear space X over the field $\mathbb{C}$ of complex numbers whose dimension is at least $k+1$. Denote by S_X and B_X the unit sphere and the unit ball of X, respectively.

Definition 1.12. A point $x \in S_X$ is called *k-extreme* of the unit ball B_X if x cannot be represented as an average of $k+1$ linearly independent elements from the unit sphere S_X. Equivalently, x is k-extreme whenever the condition $x = \frac{1}{(k+1)} \sum_{i=1}^{k+1}$, $x_i \in S_X$ for $i = 1, \dots, k+1$, implies that $x_1, x_2, ..., x_{k+1}$ are linearly dependent. Moreover, if every element of the unit sphere S_X is k-extreme, then X is called *k-rotund*.

The notion of k-extreme points was explicitly introduced in [26] and applied to theorem on uniqueness of Hahn-Banach extensions. More precisely, Zheng and Ya-Dong showed there that given at least $k+1$-dimensional normed linear space over the complex field, all bounded linear functionals defined on subspaces of X have at most k-linearly independent norm-preserving linear extensions to X if and only if the conjugate space X^* is k-rotund. In the paper [2] k-rotundity and k-extreme points found interesting application in studying the structure of nested sequences of balls in Banach spaces.

Clearly, if X is a normed space of a dimension at least l, where $l \geq k$, and $x \in S_X$ is a k-extreme point of B_X, then x is l-extreme. Moreover, 1-extreme points are just extreme points of B_X, and so 1-rotundity of X means rotundity of X.

The simple example below differentiates between k-extreme and $k+1$-extreme points.

Example 1.13. Given $k \in \mathbb{N}$, consider the $k+2$ dimensional space ℓ_1^{k+2}, equipped with ℓ_1 norm. Let $x = \left(\frac{1}{k+1}, \frac{1}{k+1}, \dots, \frac{1}{k+1}, 0\right)$. Clearly x is not a k-extreme point of $B_{\ell_1^{k+2}}$, since it can be written as an average of $k+1$ linearly independent unit vectors $e_1, e_2, \dots, e_{k+1}$. However, x is $k+1$-extreme point of $B_{\ell_1^{k+2}}$. To see it, let $x = \frac{1}{k+2} \sum_{i=1}^{k+2} y_i$, where $y_i = \left(y_i^{(1)}, y_i^{(2)}, \dots, y_i^{(k+2)}\right) \in S_{\ell_1^{k+2}}$, $i = 1, 2, \dots, k+2$.

Then

$$k+2=(k+2)\|x\|_1=\sum_{j=1}^{k+2}\sum_{i=1}^{k+2}y_i^{(j)}=\sum_{i=1}^{k+2}\sum_{j=1}^{k+2}y_i^{(j)}\le\sum_{i=1}^{k+2}\sum_{j=1}^{k+2}\left|y_i^{(j)}\right|$$
$$=\sum_{i=1}^{k+2}\|y_i\|_1=k+2.$$

Therefore $y_i^{(j)}=\left|y_i^{(j)}\right|$ for ever $i,j=1,2,\ldots,k+2$. In particular $y_i^{(k+2)}\ge 0$, $i=1,2,\ldots,k+2$, and in view of $\sum_{i=1}^{k+2}y_i^{(k+2)}=(k+2)x^{(k+2)}=0$ it follows that $y_i^{(k+2)}=0$ for every $i=1,2,\ldots,k+2$. Thus the matrix formed by vectors $y_1,y_2,\ldots,y_{k+2}$ has determinant equal to zero, since the last row comprises only of zeros. Consequently, $y_1,y_2,\ldots,y_{k+2}$ are linearly dependent, and x is $k+1$-extreme.

We wish to mention here that also the family of Orlicz sequence spaces exposes the difference between k-extreme and $k+1$-extreme points [4].

The structure of the paper is as follows. Section 2 focuses on k-extreme points in symmetric Banach function spaces. A new characterization of k-extreme points in a Banach space is given in Proposition 2.2. The main theorem of the section, Theorem 2.6, is the analogous result to Ryff's description of extreme points of orbits [21]. Part 3 considers k-extreme points in the noncommutative space $E(\mathcal{M},\tau)$. The main results of this section, Theorem 3.5 and Theorem 3.13, characterize k-extreme operators in terms of their singular value functions. They generalize the result in [6] proved for extreme points in the case of $k=1$. The closing corollary of the section relates k-rotundity of E with the k-rotundity of $E(\mathcal{M},\tau)$. In the last section of the paper we apply the obtained results to characterize k-extreme points of orbits $\Omega(g)$ and $\Omega'(g)$. Given $g\in L_1+L_\infty$ (resp. $g\in L_1[0,\alpha)$, $\alpha<\infty$), we have that f is a k-extreme point of its orbit $\Omega(g)$ (resp. $\Omega'(g)$) if and only if $\mu(f)=\mu(g)$ and $|f|\ge\mu(\infty,f)$ (resp. $\mu(f)=\mu(g)$). Therefore we obtain that k-extreme points of orbits $\Omega(g)$, and consequently of unit balls of Marcinkiewicz spaces, are non-distinguishable from extreme points.

2. k-extreme points in symmetric function spaces

Let us first state an equivalent definition of a k-extreme point in a normed space X. We will need the following simple observation, included in Lemma 1 [27].

Lemma 2.1. *If $x_1,x_2,\ldots,x_n$ are linearly dependent in B_X and $\|x_1+x_2+\cdots+x_n\|=n$, then there are complex numbers $c_1,c_2,\ldots,c_n$, not all zero, such that $c_1+c_2+\cdots+c_n=0$ and $c_1x_1+c_2x_2+\cdots+c_nx_n=0$.*

Proposition 2.2. *An element $x\in S_X$ is k-extreme of B_X if and only if whenever for the elements $u_i\in X$, $i=1,\ldots,k$, the conditions $x+u_i\in B_X$ and $x-\sum_{i=1}^k u_i\in B_X$ imply that $u_1,u_2,\ldots,u_k$ are linearly dependent.*

Proof. Suppose that $x\in S_X$ is k-extreme and $x+u_i\in B_X$ and $x-\sum_{i=1}^k u_i\in B_X$, for $u_1,u_2,\ldots,u_k\in X$. Set

$$y_i=x+u_i,\ \text{for } i=1,\ldots,k,\quad\text{and}\quad y_{k+1}=x-\sum_{i=1}^k u_i.$$

By the assumption we have $y_i \in B_X$ for $i = 1, 2, \dots, k+1$, and $\sum_{i=1}^{k+1} y_i = (k+1)x$. Consequently, $y_i \in S_X$, $i = 1, 2, \dots, k+1$, and since x is k-extreme, $y_1, y_2, \dots, y_{k+1}$ are linearly dependent. By Lemma 2.1, there exist complex numbers $c_1, c_2, \dots, c_{k+1}$, not all equal to zero, such that $c_1 + c_2 + \dots + c_{k+1} = 0$ and $c_1 y_1 + c_2 y_2 + \dots + c_{k+1} y_{k+1} = 0$. Therefore $(c_1 - c_{k+1})u_1 + (c_2 - c_{k+1})u_2 + \dots + (c_k - c_{k+1})u_k = 0$ and $u_1, u_2, \dots, u_k$ are linearly dependent, since $c_i - c_{k+1} \neq 0$ for some $i = 1, 2, \dots, k$.

Suppose now that x is not a k-extreme point of B_X, that is there exist linearly independent vectors $x_1, x_2, \dots, x_{k+1}$ from the unit sphere S_X, such that $x = \frac{1}{(k+1)} \sum_{i=1}^{k+1} x_i$. Define $u_i = x_{i+1} - x$, for $i = 1, 2, \dots, k$. Note that $u_i \neq 0$ for all $i = 1, 2, \dots, k$, since $\{x_1, x_2, \dots, x_{k+1}\}$ are linearly independent. Then $x + u_i = x_{i+1} \in B_X$, $i = 1, 2, \dots, k$, and $x - \sum_{i=1}^{k} u_i = (k+1)x - \sum_{i=2}^{k+1} x_i = x_1 \in B_X$. Moreover, it is not difficult to see that $u_1, u_2, \dots, u_k$ are linearly independent. Indeed, suppose that

$$\lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_k u_k = 0, \text{ for some } \lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{C}.$$

We have then the following equivalent equalities

$$\begin{aligned}
&\lambda_1 x_2 + \lambda_2 x_3 + \dots + \lambda_k x_{k+1} - (\lambda_1 + \lambda_2 + \dots + \lambda_k)x = 0,\\
&\lambda_1 x_2 + \dots + \lambda_k x_{k+1} - \frac{1}{k+1}(\lambda_1 + \dots + \lambda_k)(x_1 + \dots + x_{k+1}) = 0,\\
&-\frac{1}{k+1}(\lambda_1 + \dots + \lambda_k)x_1 - \frac{1}{k+1}(-k\lambda_1 + \dots + \lambda_k)x_2 - \dots\\
&-\frac{1}{k+1}(\lambda_1 + \dots - k\lambda_k)x_{k+1} = 0.
\end{aligned}$$

Since $x_1, x_2, \dots, x_{k+1}$ are linearly independent it follows that $\lambda_1 = \lambda_2 = \dots = \lambda_k = 0$, and so $u_1, u_2, \dots, u_k$ are linearly independent. □

Our main goal now is to show an analogous theorem on k-extreme points as the Ryff's theorem on extreme points [21]. We need first several preliminary results. Let us first note that E stands in this section for a symmetric Banach function space on $[0, \tau(\mathbf{1}))$, where $\tau(\mathbf{1}) \le \infty$.

Proposition 2.3. *Let f and g be decreasing, right continuous functions from the unit sphere S_E. Assume there exist points $0 < s_1 < s_2 < s_3 < s_4 < \infty$ such that*

$$f(s_i) > f(s_{i+1}), \quad i = 1, 2, 3,$$

$$\int_0^s g > \int_0^{s_1} f + f(s_1)(s - s_1), \quad \textit{for all } s_1 \le s \le s_4.$$

If $f \prec g$ then f is not a k-extreme point, for any $k \in \mathbb{N}$.

Proof. Suppose that all assumptions of the proposition are satisfied, and let

$$\epsilon = \frac{1}{k}\min\{f(s_1) - f(s_2), f(s_3) - f(s_4)\} \quad \text{and} \quad \delta = \frac{1}{2}(s_3 - s_2).$$

Define

$$u_i = -\epsilon\chi_{(s_2, s_2 + \frac{1}{k}\delta)} + \epsilon\chi_{(s_3 - \frac{i}{k}\delta, s_3 - \frac{i-1}{k}\delta)}, \quad i = 1, 2, \dots, k.$$

Set $f_i = f + u_i$, for $i = 1, 2, \dots, k$ and $f_{k+1} = f - \sum_{i=1}^{k} u_i$.

Let us show first that $f_{k+1} \prec g$. Note that

$$f_{k+1} = f + k\epsilon\chi_{(s_2, s_2 + \frac{1}{k}\delta)} - \epsilon\chi_{(s_3 - \delta, s_3)},$$

and so for $0 \le s \le s_1$ and $s \ge s_4$, $\mu(s, f_{k+1}) = f(s)$. Hence taking $s_1 \le s \le s_4$ we have that $\mu(s, f_{k+1}) \le f(s_1)$ and

$$\int_0^s \mu(f_{k+1}) = \int_0^{s_1} f + \int_{s_1}^s \mu(f_{k+1}) \le \int_0^{s_1} f + f(s_1)(s - s_1) < \int_0^s g,$$

$$\int_{s_1}^{s_4} \mu(f_{k+1}) = \int_{s_1}^{s_4} f_{k+1} = \int_{s_1}^{s_4} f + k\epsilon \cdot \frac{1}{k}\delta - \epsilon\delta = \int_{s_1}^{s_4} f.$$

Combining the previous equality with the fact $\mu(s, f_{k+1}) = f(s)$ for $s \ge s_4$ and $0 < s \le s_1$, it follows that for $s \ge s_4$,

$$\begin{aligned}\int_0^s \mu(f_{k+1}) &= \int_0^{s_1} \mu(f_{k+1}) + \int_{s_1}^{s_4} \mu(f_{k+1}) + \int_{s_4}^s \mu(f_{k+1}) \\ &= \int_0^{s_4} f + \int_{s_4}^s f = \int_0^s f.\end{aligned}$$

Therefore $f_{k+1} \prec g$ and consequently $f_{k+1} \in B_E$. Similarly one can show that $f_i \prec g$ and so $f_i \in B_E$ for all $i = 1, 2, \dots, k$.

Since $u_1, u_2, \dots, u_k$ are linearly independent, by Proposition 2.2, f cannot be a k-extreme point. □

Lemma 2.4. *Let f and g be decreasing, right continuous functions from the unit sphere S_E, such that $f \prec g$. If for some $t_0 > 0$, f is not constant in any of its right neighborhoods, and*

$$\int_0^{t_0} f < \int_0^{t_0} g,$$

then f is not a k-extreme point of B_E.

Proof. Let $\eta > 0$ be such that

$$\int_0^{t_0} f + \eta < \int_0^{t_0} g.$$

Since f has infinitely many values on every right neighborhood of t_0, we can chose $t_0 < s_1 < s_2 < s_3 < s_4 < t_0 + \eta/(2f(t_0))$ such that $f(s_1) > f(s_2) > f(s_3) > f(s_4)$.

Since $s_1 - t_0 < \eta/(2f(t_0))$, taking $s \ge t_0$ we have that

$$\begin{aligned}\int_0^{s_1} f + \eta &= \int_0^{t_0} f + \eta + \int_{t_0}^{s_1} f < \int_0^{t_0} g + \int_{t_0}^{s_1} f \\ &\le \int_0^{t_0} g + f(t_0)(s_1 - t_0) \le \int_0^{t_0} g + \frac{\eta}{2} \le \int_0^s g + \frac{\eta}{2}.\end{aligned}$$

Consequently for $s \ge t_0$,

$$\int_0^{s_1} f + \frac{\eta}{2} < \int_0^s g.$$

Notice that

$$0 < s_4 - s_1 \le \frac{\eta}{2f(t_0)} \le \frac{\eta}{2f(s_1)}.$$

Hence for $s_1 \le s \le s_4$,

$$\int_0^{s_1} f + f(s_1)(s - s_1) \le \int_0^{s_1} f + f(s_1)(s_4 - s_1) \le \int_0^{s_1} f + \frac{\eta}{2} < \int_0^s g,$$

and by Proposition 2.3, f cannot be a k-extreme point of B_E. □

Lemma 2.5. *Let f and g be decreasing, right continuous functions from the unit sphere S_E, such that $f \prec g$. If for some $t_0 > 0$, f is continuous at t_0, not constant in any of its left neighborhoods and*

$$f(t_0) < g(t_0),$$

then f is not a k-extreme point of B_E.

Proof. By continuity of f at t_0 and the assumption that $f(t_0) < g(t_0)$ there exist $\delta, \eta_1 > 0$ such that $f(t) + \eta_1 < g(t)$ for all $t \in [t_0 - \delta, t_0]$. Hence $\int_0^{t_0} f + \eta < \int_0^{t_0} g$, for some $\eta > 0$.

Set $C = \min\{\delta, \eta/(2g(t_0 - \delta))\}$. Note that $g(t_0 - \delta) > 0$, and so C is well defined.

The assumptions on the function f ensure that f has infinitely many values on every left neighborhood of t_0. Thus we can find $t_0 - C < s_1 < s_2 < s_3 < s_4 < t_0$, such that $f(s_1) > f(s_2) > f(s_3) > f(s_4)$. Note that $t_0 - s_1 < C$, and so $t_0 - s_1 < \eta/(2g(t_0 - \delta))$ and $t_0 - \delta < s_1$. Thus for all $s \in [s_1, t_0]$,

$$\begin{aligned}\int_0^{s_1} f + \eta \le \int_0^{t_0} f + \eta < \int_0^{t_0} g = \int_0^{s_1} g + \int_{s_1}^{t_0} g \le \int_0^{s_1} g + g(s_1)(t_0 - s_1) \\ \le \int_0^{s_1} g + g(t_0 - \delta)\frac{\eta}{2g(t_0 - \delta)} < \int_0^{s_1} g + \frac{\eta}{2} < \int_0^{s} g + \frac{\eta}{2}.\end{aligned}$$

Moreover $0 < s_4 - s_1 < \eta/(2g(t_0 - \delta)) \le \eta/(2f(t_0 - \delta)) \le \eta/(2f(s_1))$. Consequently, for all $s_1 \le s \le t_0$, we have that

$$\int_0^{s_1} f + f(s_1)(s - s_1) \le \int_0^{s_1} f + f(s_1)(s_4 - s_1) \le \int_0^{s_1} f + \frac{\eta}{2} < \int_0^{s} g.$$

It follows now by Proposition 2.3 that f is not a k-extreme point of B_E. $\square$

Theorem 2.6. *Let E be a symmetric Banach function space and $f \in S_E$. Suppose there exists a function $g \in S_E$ such that $f \prec g$ and $\mu(f) \neq \mu(g)$. Then $\mu(f)$ cannot be a k-extreme point of B_E.*

Proof. Since the condition $f \prec g$ is equivalent with $\mu(f) \prec \mu(g)$, we can assume that $f = \mu(f)$ and $g = \mu(g)$. Thus f and g are decreasing, right continuous functions. If $f \neq g$, then for at least one value of $t > 0$,

$$\int_0^t f < \int_0^t g.$$

Then the set $A = \{u > 0 : f(u) < g(u)\}$ contains a point at which f is continuous. Indeed, the condition that $f \neq g$ implies that A is non-empty. Furthermore, since g is right continuous and f is decreasing, one can show that the set A contains a non-empty interval. Since a decreasing function f has only countably many points of discontinuity, the claim follows.

Let $t_0 > 0$ be such that $f(t_0) < g(t_0)$ and f be continuous at t_0.

Case I Assume that $t_0 \in (t_1, t_2)$, $0 \le t_1 < t_2 < \infty$, $f = c$ on $[t_1, t_2)$, and f experiences a jump discontinuities greater than η at t_1 and t_2. If $t_1 = 0$, disregard the discontinuity at t_1. Note that since $f(t_0) < g(t_0)$, we have that $c = f(t) < g(t)$ for all $t \in [t_1, t_0]$. Let $\alpha = \sup\{t_1 \le t \le t_2 : f(t) < g(t)\}$. Clearly $\alpha > t_0$.

Case I.a Assume that $\alpha = t_2$ or $g(t_2^-) \ge c$. Let

$$\delta < \frac{t_0 - t_1}{2k} \quad \text{and} \quad \epsilon < \frac{1}{k}\min\{g(t_0) - c, \eta\}.$$

For $i = 1, 2, \dots, k$, define

$$\begin{aligned} u_i &= -\epsilon\chi_{(t_1,t_1+\delta)} + \epsilon\chi_{(t_2-i\delta,t_2-(i-1)\delta)} \\ f_i &= f + u_i \\ f_{k+1} &= f - \sum_{i=1}^{k} u_i = f + k\epsilon\chi_{(t_1,t_1+\delta)} - \epsilon\chi_{(t_2-k\delta,t_2)}. \end{aligned}$$

It is clear that $\mu(f_{k+1}) = f_{k+1}$. We will show next that $f_{k+1} \prec g$. For $t \le t_1$, $\int_0^t f_{k+1} = \int_0^t f \le \int_0^t g$. Observe that $c = f\chi_{[t_1,t_2)} \le g\chi_{[t_1,t_2)}$, and so $f_{k+1}\chi_{[t_1,t_2)} \le g\chi_{[t_1,t_2)}$. Thus taking $t_1 < t \le t_2$, it follows that $\int_0^t f_{k+1} = \int_0^{t_1} f + \int_{t_1}^t f_{k+1} \le \int_0^{t_1} f + \int_{t_1}^t g \le \int_0^t g$. Finally if $t > t_2$, $\int_0^t f_{k+1} = \int_0^t f + k\epsilon\delta - k\epsilon\delta = \int_0^t f \le \int_0^t g$.

Note next that $\mu(f_i) = f + \epsilon\chi_{(t_1,t_1+\delta)} - \epsilon\chi_{(t_2-\delta,t_2)}$, for all $i = 1, 2, \dots, k$. Following the same argument as above, one can show that $f_i \prec g$, $i = 1, 2, \dots, k$. By linear independence of $u_1, u_2, \dots, u_k$ and by Proposition 2.2, f is not a k-extreme point of B_E.

Case I.b Let now $\alpha = \sup\{t_1 \le t \le t_2 : f(t) < g(t)\} < t_2$ and $g(t_2^-) < c$. Then for γ, β sufficiently small $g(t) < c - \beta$, whenever $t_2 - \gamma \le t < t_2$. Set

$$\delta < \frac{1}{k}\min\left\{\frac{t_0 - t_1}{2}, t_2 - \alpha, \alpha - t_1, \gamma\right\} \quad \text{and} \quad \epsilon < \frac{1}{k}\min\{g(t_0) - c, \eta, \beta\}.$$

Define functions u_i and f_i as in Case I.a.

We will show first that $f_{k+1} \prec g$, where obviously $\mu(f_{k+1}) = f_{k+1}$. Since for $t \le t_1$, $f_{k+1}(t) = f(t)$ and for $t_1 < t < \alpha$, $f_{k+1}(t) \le g(t)$, it remains to consider the case of $t \ge \alpha$. Let $\alpha \le t \le t_2$. By $g\chi_{[\alpha,t_2]} \le f_{k+1}\chi_{[\alpha,t_2]}$ and $\int_0^{t_2} f_{k+1} = \int_0^{t_2} f$, it follows that

$$\int_0^t f_{k+1} = \int_0^{t_2} f - \int_t^{t_2} f_{k+1} \le \int_0^{t_2} f - \int_t^{t_2} g \le \int_0^{t_2} g - \int_t^{t_2} g = \int_0^t g.$$

For $t > t_2$, $\int_0^t f_{k+1} = \int_0^t f \le \int_0^t g$.

Since $\mu(f_i) = f + \epsilon\chi_{(t_1,t_1+\delta)} - \epsilon\chi_{(t_2-\delta,t_2)}$, for all $i = 1, 2, \dots, k$, following the similar argument as above it is not difficult to observe that $f_i \prec g$, $i = 1, 2, \dots, k$. Again by Proposition 2.2, f is not a k-extreme point of B_E.

Case II Suppose now that $t_0 \in (t_1, t_2)$, $0 < t_1 < t_2 \le \infty$, $f = c$ on $[t_1, t_2)$, and f is continuous at t_1, where $t_1 = \inf\{t : f(t) = c\}$. Since $f(t_0) < g(t_0)$, we have that $f(t_1) < g(t_1)$. By Lemma 2.5 applied to t_1 we can conclude now that f is not a k-extreme point of B_E.

Case III Assume that $t_0 \in (t_1, t_2)$, $0 \le t_1 < t_2 < \infty$, $f = c$ on $[t_1, t_2)$, and f is continuous at t_2, where $t_2 = \sup\{t : f(t) = c\}$. We claim that $\int_0^{t_2} f < \int_0^{t_2} g$ and so by Lemma 2.4, f is not k-extreme. Indeed, suppose that $\int_0^{t_2} f = \int_0^{t_2} g$.

If $g(t_2^-) \ge f(t_2)$, by the continuity of f at t_0, and inequality $f(t_0) < g(t_0)$, we have $\int_{t_0}^{t_2} f < \int_{t_0}^{t_2} g$. Hence

$$\int_0^{t_0} f = \int_0^{t_2} f - \int_{t_0}^{t_2} f = \int_0^{t_2} g - \int_{t_0}^{t_2} f > \int_0^{t_2} g - \int_{t_0}^{t_2} g = \int_0^{t_0} g,$$

contradicting the assumption that $f \prec g$.

On the other hand if $g(t_2^-) < f(t_2)$, there exists $\delta > 0$ such that $g(t) < f(t)$ for $t \in (t_2, t_2 + \delta)$. Then for $t \in (t_2, t_2 + \delta)$,

$$\int_0^t f = \int_0^{t_2} f + \int_{t_2}^t f = \int_0^{t_2} g + \int_{t_2}^t f > \int_0^{t_2} g + \int_{t_2}^t g = \int_0^t g,$$

which leads to a contradiction. Hence $\int_0^{t_2} f < \int_0^{t_2} g$, where f is not constant in any of the right neighborhoods of t_2. By Lemma 2.4 it follows that f is not a k-extreme point.

Case IV Suppose now that $t_0 \in (t_1, \infty)$, where $f = c$ on $[t_1, \infty)$, and $t_1 = \inf\{t : f(t) = c\} \geq 0$. Moreover, assume that f has a jump discontinuity greater than η at t_1, if $t_1 > 0$. Note first that for $t \geq t_1$,

$$\int_0^{t_1} f + c(t - t_1) = \int_0^t f \leq \int_0^t g, \tag{2.1}$$

and so

$$\frac{\int_0^{t_1} f}{t - t_1} + c \leq \frac{\int_0^t g}{t - t_1}. \tag{2.2}$$

It is clear that $\lim_{t\to\infty} \frac{\int_0^{t_1} f}{t - t_1} = 0$. Moreover, by (2.1) it is easy to see that $\lim_{t\to\infty} \int_0^t g = \infty$. Hence $\lim\limits_{t\to\infty} \dfrac{\int_0^t g}{t - t_1} = \lim\limits_{t\to\infty} g(t)$. Consequently by (2.2), $\lim_{t\to\infty} g(t) \geq c$. Set

$$\epsilon < \min\{\eta, g(t_0) - f(t_0)\}, \quad \text{and} \quad \delta = \frac{t_0 - t_1}{k}.$$

If $t_1 = 0$ disregard η in the inequality above. Define the functions
$u_i = -\epsilon\chi_{(t_0 - i\delta, t_0 - (i-1)\delta)}$, $f_i = f + u_i$, for $i = 1, 2, \ldots, k$, and $f_{k+1} = f - \sum_{i=1}^k u_i$.

Consider first the decreasing function $f_{k+1} = f + \epsilon\chi_{(t_1, t_0)}$. We have that $f_{k+1}(t) = f(t) + \epsilon\chi_{(t_1,t_0)}(t) \leq g(t)$ for all $t \geq t_1$, and $f_{k+1}(t) = f(t)$ for $0 < t < t_1$. It is easy to observe now that $f_{k+1} \prec g$.

Moreover, for $i = 1, 2, \ldots, k$, $\mu(f_i) = f$. In view of Proposition 2.2, the claim follows.

Case V Consider now the case when f differs from a constant in every neighbourhood of t_0. Since f is continuous at t_0 and $f(t_0) < g(t_0)$, Lemma 2.5 ensures now that f is not a k-extreme point. $\square$

Corollary 2.7. *Let E be a symmetric Banach function space and $f \in S_E$. If $\mu(f)$ is a k-extreme point of B_E then for all functions $g \in S_E$ with $f \prec g$, it holds that $\mu(f) = \mu(g)$.*

We wish to observe that the same characterization of the k-extreme points is not valid for symmetric sequence spaces. We are grateful to Timur Oikhberg for bringing it to our attention and providing the following example.

Consider the points $x = (\frac{1}{2}, \frac{1}{2}, 0)$ and $y = (1, 0, 0)$ in ℓ_1. It is easy to verify that x is a 2-extreme point in ℓ_1 with $x \prec y$. However $x = \mu(x) \neq \mu(y) = y$.

3. k-extreme points in noncommutative Spaces

In this section we shall characterize k-extreme points of the ball $B_{E(\mathcal{M},\tau)}$ in terms of their singular value functions. Through the effort of the series of technical lemmas we will establish main Theorems 3.5 and 3.13.

Lemma 3.1. *Let $\mathcal{M}$ be non-atomic and x be a non-zero, positive element of $S(\mathcal{M},\tau)$. Then there exist k mutually orthogonal, non-zero projections $p_1, p_2, \dots, p_k$, commuting with x and such that $p_i \le s(x)$, $i = 1, 2, \dots, k$.*

Proof. Suppose first that $\mu(x)$ admits at least $k+1$ different values. Choose $0 < \lambda_1 < \lambda_2 < \dots < \lambda_{k+1}$ such that $\mu(\lambda_1, x) > \mu(\lambda_2, x) > \dots > \mu(\lambda_{k+1}, x) \ge \mu(\infty, x)$. Since $\tau(e^x(\lambda,\infty)) < \infty$ for all $\lambda > \mu(\infty, x)$, we have that

$$\tau\left(e^x\left(\mu(\lambda_{i+1},x),\mu(\lambda_i,x)\right]\right) = m\{t : \mu(\lambda_{i+1},x) < \mu(t,x) \le \mu(\lambda_i,x)\} > 0,$$

$i = 1, 2, \dots, k$. Hence $0 \ne p_i = e^x\left(\mu(\lambda_{i+1},x),\mu(\lambda_i,x)\right] \le e^x(0,\infty) = s(x)$, $i = 1, 2, \dots, k$. Clearly $p_1, p_2, \dots, p_k$ are mutually orthogonal projections commuting with x.

Assume now that $\mu(x)$ has less than $k+1$ different values. Then $\mu(0,x) < \infty$ is the biggest value of $\mu(x)$. Moreover $\tau\left(e^x[\mu(0,x),\infty)\right) = m\{t : \mu(t,x) \ge \mu(0,x)\} > 0$, if $\mu(0,x) > \mu(\infty,x)$ or $e^x[\mu(0,x),\infty) = \mathbf{1}$, if $\mu(0,x) = \mu(\infty,x)$. In either case $e^x[\mu(0,x),\infty)$ is a non-zero projection less than $s(x) = e^x(0,\infty)$.

Since $\mathcal{M}$ is non-atomic, we can find k mutually orthogonal, non-zero projections $p_1, p_2, \dots, p_k$ such that $p_i \le e^x[\mu(0,x),\infty)$, $i = 1, 2, \dots, k$. We claim that p_i, $i = 1, 2, \dots, k$, commute with all spectral projections of the form $e^x(s,\infty)$, $s > 0$. Indeed, if $s \ge \mu(0,x)$, then $e^x(s,\infty) = 0$. For $s < \mu(0,x)$, $e^x[\mu(0,x),\infty) \le e^x(s,\infty)$ and so $p_i \le e^x(s,\infty)$, $i = 1, 2, \dots, k$. Thus if $s < \mu(0,x)$, $p_i e^x(s,\infty) = p_i = e^x(s,\infty)p_i$, $i = 1, 2, \dots, k$. Proposition 1.1 implies now that all projections $p_1, p_2, \dots, p_k$ commute with x. □

Lemma 3.2. *Let $\mathcal{M}$ be a non-atomic von Neumann algebra. If x is a k-extreme point of the unit ball $\mathrm{B}_{E(\mathcal{M},\tau)}$ then $\mu(\infty,x) = 0$ or $n(x)\mathcal{M}n(x^*) = 0$.*

Proof. Assume for a contrary that $n(x)\mathcal{M}n(x^*) \ne 0$ and $\mu(\infty,x) > 0$, while x is a k-extreme point. As shown in Lemma 2.6 [9], if $n(x)\mathcal{M}n(x^*) \ne 0$, there exists an isometry w such that $x = w\,|x|$.

It is easy to show now that if x is k-extreme then so is $|x|$. Indeed, let $|x| + u_i \in B_{E(\mathcal{M},\tau)}$, for $i = 1, 2, \dots, k$ and $|x| - \sum_{i=1}^k u_i \in B_{E(\mathcal{M},\tau)}$. Then $x + wu_i \in B_{E(\mathcal{M},\tau)}$, $i = 1, 2, \dots, k$ and $x - \sum_{i=1}^k wu_i \in B_{E(\mathcal{M},\tau)}$, and since x is k-extreme $wu_1, wu_2, \dots, wu_k$ are linearly dependent. In view of w being an isometry, $u_1, u_2, \dots, u_k$ are linearly dependent, proving that $|x|$ is k-extreme.

Since $n(x) \ne 0$ and $\mathcal{M}$ is non-atomic, there exist k mutually orthogonal, non-zero projections, $p_1, p_2, \dots, p_k$, such that $p_i \le n(x)$, $i = 1, 2, \dots, k$. By Corollary 1.6, $\mu(|x| + \mu(\infty,x)p_i) = \mu(x)$, and $\mu(|x| - \sum_{i=1}^k \mu(\infty,x)p_i) = \mu(|x| + \sum_{i=1}^k \mu(\infty,x)p_i) = \mu(x)$. Clearly the set

$$\{\mu(\infty,x)p_1, \mu(\infty,x)p_2, \dots, \mu(\infty,x)p_k\}$$

is linearly independent, which is impossible since $|x|$ is k-extreme. □

Lemma 3.3. *Let $\mathcal{M}$ be a non-atomic von Neumann algebra. If x is a k-extreme point of $B_{E(\mathcal{M},\tau)}$ then $|x| \ge \mu(\infty,x)s(x)$.*

Proof. Suppose that $\mu(\infty,x) > 0$ and $e^{|x|}(0,\mu(\infty,x)) \ne 0$. We have that $|x|\,e^{|x|}(0,\mu(\infty,x)) \ne 0$, since $e^{|x|}(0,\mu(\infty,x)) \le s(x) = e^{|x|}(0,\infty)$.

Choose $0 < \epsilon < 1$ such that $e^{|x|}(0,\beta] \ne 0$, where $\beta = \frac{1}{1+\epsilon}\mu(\infty,x)$. Such ϵ must exist, since otherwise $e^{|x|}(0,\mu(\infty,x)) = 0$.

By Lemma 3.1 applied to $|x|\, e^{|x|}(0,\beta]$, we can find k non-zero, mutually orthogonal projections $\{p_1,p_2,\ldots,p_k\}$, $p_i \le e^{|x|}(0,\beta]$ and commuting with $|x|\, e^{|x|}(0,\beta]$. Consequently p_i commute with $|x|$ for all $i=1,2,\ldots,k$. Indeed, since $p_i e^{|x|}(\beta,\infty) = e^{|x|}(\beta,\infty)p_i = 0$ and p_i commutes with $|x|\, e^{|x|}[0,\beta]$, we have that for all $i = 1,2,\ldots,k$,

$$\begin{aligned}
|x|\, p_i &= |x|\, e^{|x|}\{0\}p_i + |x|\, e^{|x|}(0,\beta]p_i + |x|\, e^{|x|}(\beta,\infty)p_i = |x|\, e^{|x|}(0,\beta]p_i \\
&= p_i\, |x|\, e^{|x|}\{0\} + p_i\, |x|\, e^{|x|}(0,\beta] + p_i e^{|x|}(\beta,\infty)\, |x| \\
&= p_i\, |x|\, e^{|x|}\{0\} + p_i\, |x|\, e^{|x|}(0,\beta] + p_i\, |x|\, e^{|x|}(\beta,\infty) = p_i\, |x|\,.
\end{aligned}$$

Define $u_i = -\epsilon\, |x|\, p_i$, $z_i = |x| + u_i = |x| - \epsilon\, |x|\, p_i$, $i=1,2,\ldots,k$, and $z_{k+1} = |x| - \sum_{i=1}^k u_i = |x| + \epsilon\, |x| \sum_{i=1}^k p_i$. For $i=1,2,\ldots,k$, $0 \le |x| - |x|\, p_i \le z_i \le |x|$, and so $z_i \in B_{E(\mathcal{M},\tau)}$. Furthermore, it was shown in Lemma 3.8 [9], that $|x| + \epsilon\, |x|\, e^{|x|}(0,\beta] \in B_{E(\mathcal{M},\tau)}$. Since $z_{k+1} \le |x| + \epsilon\, |x|\, e^{|x|}(0,\beta]$ it follows that also $z_{k+1} \in B_{E(\mathcal{M},\tau)}$.

Let $x = u\, |x|$ be the polar decomposition of x. Therefore $x - \epsilon x p_i = u z_i \in B_{E(\mathcal{M},\tau)}$, for $i=1,2,\ldots,k$ and $x + \epsilon x \sum_{i=1}^k p_i = u z_{k+1} \in B_{E(\mathcal{M},\tau)}$. Note that for any non-zero projection $q \le e^{|x|}(0,\beta] \le s(x)$ we have that $xq \ne 0$. Therefore the collection $\{-\epsilon x p_i\}_{i=1}^k$ is linearly independent, and x cannot be a k-extreme point.

Therefore $e^{|x|}(0,\beta] = 0$ for all $\beta < \mu(\infty,x)$. Hence $e^{|x|}(0,\mu(\infty,x)) = 0$ and $s(x) = e^{|x|}(0,\infty) = e^{|x|}[\mu(\infty,x),\infty)$. Consequently,

$$|x| = \int_{[\mu(\infty,x),\infty)} \lambda de^{|x|}(\lambda) \ge \mu(\infty,x)e^{|x|}[\mu(\infty,x),\infty) = \mu(\infty,x)s(x).$$

□

The inequality $|x| \ge \mu(\infty,x)s(x)$ can be expressed in an equivalent way as follows.

Lemma 3.4. *Let $x \in S(\mathcal{M},\tau)$ and $\mu(\infty,x) > 0$. Then the conditions $|x| \ge \mu(\infty,x)s(x)$ and $e^{|x|}(0,\mu(\infty,x)) = 0$ are equivalent.*

Proof. It follows from the last paragraph of the proof of Lemma 3.3 that if $e^{|x|}(0,\mu(\infty,x)) = 0$ then $|x| \ge \mu(\infty,x)s(x)$.

For the converse, assume that $|x| \ge \mu(\infty,x)s(x)$. Since for any $\alpha > 0$, $e^{|x|}(0,\alpha] \le s(x)$ and $|x|$ commutes with $e^{|x|}(0,\alpha]$, we have that $|x|\, e^{|x|}(0,\alpha] \ge \mu(\infty,x)e^{|x|}(0,\alpha]$. Suppose that there exists $0 < \alpha < \mu(\infty,x)$ such that $e^{|x|}(0,\alpha] \ne 0$. Then

$$\begin{aligned}
|x|\, e^{|x|}(0,\alpha] &= \int_{(0,\alpha]} \lambda de^{|x|}(\lambda) \le \alpha \int_{(0,\alpha]} de^{|x|}(\lambda) = \alpha e^{|x|}(0,\alpha] \\
&\le \mu(\infty,x)e^{|x|}(0,\alpha],
\end{aligned}$$

and $\alpha e^{|x|}(0,\alpha] \ne \mu(\infty,x)e^{|x|}(0,\alpha]$. Hence for all $\alpha < \mu(\infty,x)$, $e^{|x|}(0,\alpha] = 0$. Therefore $e^{|x|}(0,\mu(\infty,x)) = \vee_{0<\alpha<\mu(\infty,x)} e^{|x|}(0,\alpha] = 0$. □

Theorem 3.5. *Suppose that $\mathcal{M}$ is a non-atomic von Neumann algebra with a σ-finite trace τ. If x is a k-extreme point of $B_{E(\mathcal{M},\tau)}$ then $\mu(x)$ is a k-extreme point of B_E and either*

(i) *$\mu(\infty,x) = 0$ or*

(ii) *$n(x)\mathcal{M}n(x^*) = 0$ and $|x| \ge \mu(\infty,x)s(x)$.*

Proof. Suppose that x is a k-extreme point of the unit ball in $E(\mathcal{M},\tau)$. By Lemma 3.2 and 3.3 conditions (i) or (ii) are satisfied.

Let

$$\mu(x)=\frac{1}{k+1}\sum_{i=1}^{k+1} f_i,\ \text{ where } f_i\in S_E,\ i=1,2,\dots,k+1. \tag{3.1}$$

To prove that $\mu(x)$ is k-extreme we need to show that $f_1,f_2,\dots,f_{k+1}$ are linearly dependent. Let

$$p=s(|x|-\mu(\infty,x)s(x))=e^{|x|}(\mu(\infty,x),\infty).$$

By Corollary 1.10, there exist a projection $q\in\mathcal{P}(\mathcal{M})$, a non-atomic commutative von Neumann subalgebra $\mathcal{N}\subset q\mathcal{M}q$ and a $*$-isomorphism V acting from the $*$-algebra $S\left([0,\tau(\mathbf{1})),m\right)$ into the $*$-algebra $S(\mathcal{N},\tau)$, such that

$$V\mu(x)=|x|\,q \quad\text{ and }\quad \mu(V(f))=\mu(f)\quad\text{for all } f\in S\left([0,\tau(\mathbf{1})),m\right).$$

Moreover, there are three choices of q: (1) $q=\mathbf{1}$ whenever $\tau(s(x))<\infty$, (2) $q=s(x)$ if $\tau(s(x))=\infty$ and $\tau(p)<\infty$, or (3) $q=p$ if $\tau(p)=\infty$.

Applying now isomorphism V to the equation (3.1),

$$|x|\,q=V\mu(x)=\frac{1}{k+1}\sum_{i=1}^{k+1}V(f_i). \tag{3.2}$$

Case (1). Let $\tau(s(x))<\infty$ and $q=\mathbf{1}$. Since $s(x)\sim s(x^*)$ and $\tau(s(x))<\infty$, by [25, Chapter 5, Proposition 1.38] $n(x)\sim n(x^*)$. Then by [9, Lemma 2.6] there exists an isometry w such that $x=w\,|x|$. Therefore by (3.2) we have

$$x=\frac{1}{k+1}\sum_{i=1}^{k+1}wV(f_i),$$

and $wV(f_1),wV(f_2),\dots,wV(f_{k+1})$ are linearly dependent. Since w and V are isometries $f_1,f_2,\dots,f_{k+1}$ are linearly dependent.

Case (2). Suppose that $\tau(s(x))=\infty$, $\tau(p)<\infty$, and $q=s(x)$. Let $x=u\,|x|$ be the polar decomposition of x. By (3.2)

$$x=\frac{1}{k+1}\sum_{i=1}^{k+1}uV(f_i),$$

where $uV(f_i)\in B_{E(\mathcal{M},\tau)}$, $i=1,2,\dots,k+1$. Since x is k-extreme there exist constants $C_1,C_2,\dots,C_{k+1}$, such that $\sum_{i=1}^{k+1}C_i\neq 0$ and $\sum_{i=1}^{k+1}C_iuV(f_i)=0$. However $q=s(x)$ is an identity in the von Neumann algebra $\mathcal{N}\subset s(x)\mathcal{M}s(x)$ and so $u^*uV(f_i)=s(x)V(f_i)=V(f_i)$. Consequently,

$$\sum_{i=1}^{k+1}C_iV(f_i)=0$$

and since V is injective $f_1,f_2,\dots,f_{k+1}$ are linearly dependent.

Case (3). Consider now the case when $q=p=e^{|x|}(\mu(\infty,x),\infty)$ and $\tau(p)=\infty$. Thus in view of Lemma 3.4, $q^{\perp}=e^{|x|}\{0\}+e^{|x|}\{\mu(\infty,x)\}\geq e^{|x|}\{\mu(\infty,x)\}$.

For each $i = 1, 2, \dots, k+1$, choose $0 \le \alpha_i \le \mu(\infty, f_i)$ such that $\frac{1}{k+1}\sum_{i=1}^{k+1}\alpha_i = \mu(\infty, x)$. Such constants exist, since by [14, Lemma 2.5] for all $t > 0$

$$\mu(t, x) = \mu\left(t, \frac{1}{k+1}\sum_{i=1}^{k+1} fi\right) \le \frac{1}{k+1}\sum_{i=1}^{k+1}\mu\left(\frac{t}{k+1}, f_i\right),$$

and so $\mu(\infty, x) \le \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(\infty, f_i)$.

Define operators $x_i = V(f_i) + \alpha_i e^{|x|}\{\mu(\infty, x)\}$. Observe that since q is an identity in $\mathcal{N}$, $q^\perp V(f_i) = V(f_i)q^\perp = 0$, and so $e^{|x|}\{\mu(\infty, x)\}V(f_i) = V(f_i)e^{|x|}\{\mu(\infty, x)\} = 0$. Furthermore $\alpha_i \le \mu(\infty, f_i) = \mu(\infty, V(f_i))$, and hence by Corollary 1.6, $\mu(x_i) = \mu(V(f_i)) = \mu(f_i)$. Hence $x_i \in B_{E(\mathcal{M},\tau)}$ for all $i = 1, 2, \dots, k+1$. We have now by (3.2) that

$$\begin{aligned} |x| &= |x|\, q + |x|\, e^{|x|}\{\mu(\infty, x)\} = |x|\, q + \mu(\infty, x)e^{|x|}\{\mu(\infty, x)\} \\ &= \frac{1}{k+1}\sum_{i=1}^{k+1} V(f_i) + \frac{1}{k+1}\sum_{i=1}^{k+1}\alpha_i e^{|x|}\{\mu(\infty, x)\} = \frac{1}{k+1}\sum_{i=1}^{k+1} x_i. \end{aligned}$$

Using the polar decomposition $x = u\,|x|$,

$$x = \frac{1}{k+1}\sum_{i=1}^{k+1} u x_i = \frac{1}{k+1}\sum_{i=1}^{k+1}(uV(f_i) + \alpha_i u e^{|x|}\{\mu(\infty, x)\}),$$

and $ux_1, ux_2, \dots, ux_{k+1}$ are linearly dependent. Since two components of x_i, $uV(f_i)$ and $\alpha_i u e^{|x|}\{\mu(\infty, x)\}$ have disjoint supports, $uV(f_1), \dots, uV(f_{k+1})$ are linearly dependent. Moreover $q \le s(x)$, and so $u^*uV(f_i) = s(x)V(f_i) = s(x)qV(f_i) = qV(f_i) = V(f_i)$. Since V is an isometry, $f_1, f_2, \dots, f_{k+1}$ are linearly dependent. □

In order to show the converse statement of Theorem 3.5 we need several preliminary results.

Lemma 3.6. *Let $x \in S_{E(\mathcal{M},\tau)}$ and $\mu(x)$ be a k-extreme point of B_E. If $x = \frac{1}{k+1}\sum_{i=1}^{k+1} x_i$, where $x_i \in B_{E(\mathcal{M},\tau)}$, then $\mu(x) = \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(x_i)$.*

Proof. Let $x \in S_{E(\mathcal{M},\tau)}$ and $x = \frac{1}{k+1}\sum_{i=1}^{k+1} x_i$, where $x_i \in B_{E(\mathcal{M},\tau)}$. Since $\mu(x) \prec \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(x_i)$, where $\frac{1}{k+1}\sum_{i=1}^{k+1}\mu(x_i) \in B_E$, the claim follows by Corollary 2.7. □

Proposition 3.7. *Let $x \in S_{E(\mathcal{M},\tau)}$ be such that $\mu(x)$ is a k-extreme point of B_E, $\mu(\infty, x) > 0$ and $x \ge \mu(\infty, x)s(x)$. Let $x = \frac{1}{k+1}\sum_{i=1}^{k+1} x_i$, where $x_i \in S_{E(\mathcal{M},\tau)}$, $x_i \ge 0$, $i = 1, 2, \dots, k+1$. Then for every $i = 1, 2, \dots, k+1$, $x_i \ge \mu(\infty, x_i)s(x_i)$. Moreover, if for some i, $\mu(\infty, x_i) > 0$ then $s(x_i) = s(x)$.*

Proof. Let $\mu(x)$ be k-extreme in E, $\mu(\infty, x) > 0$, $x \ge \mu(\infty, x)s(x)$, and $x = \frac{1}{k+1}\sum_{i=1}^{k+1} x_i$, where $x_i \in S_{E(\mathcal{M},\tau)}$, $x_i \ge 0$, for $i = 1, 2, \dots, k+1$. By the assumption $\mu(\infty, x) > 0$, $\tau(\mathbf{1}) = \infty$.

Observe first that we can assume without loss of generality that $\mathcal{M}$ is non-atomic. If not, we can consider new elements $\mathbb{1}\overline{\otimes}x, \mathbb{1}\overline{\otimes}x_i \in S_{E(\mathcal{A},\kappa)}$, $i = 1, 2, \dots, k+1$, where $\mathcal{A} = \mathcal{N} \otimes \mathcal{M}$ is a non-atomic von Neumann algebra (see Remark 1.11) and $\mathbb{1}\overline{\otimes}x = \frac{1}{k+1}\sum_{i=1}^{k+1}\mathbb{1}\overline{\otimes}x_i$. Observe that $\tilde{\mu}(\infty, \mathbb{1}\overline{\otimes}x) = \mu(\infty, x)$ and $\tilde{\mu}(\infty, \mathbb{1}\overline{\otimes}x_i) = \mu(\infty, x_i)$. Moreover, $s(x) = s(x_i)$ if and only if $s(\mathbb{1}\overline{\otimes}x) = \mathbb{1} \otimes s(x) = \mathbb{1} \otimes s(x_i) = s(\mathbb{1}\overline{\otimes}x_i)$. Finally, $\mathbb{1}\overline{\otimes}x - \tilde{\mu}(\infty, \mathbb{1}\overline{\otimes}x)s(\mathbb{1}\overline{\otimes}x) = \mathbb{1}\overline{\otimes}x - \mu(\infty, x)\mathbb{1} \otimes s(x) = \mathbb{1}\overline{\otimes}(x - \mu(\infty, x)s(x))$,

and so $x \geq \mu(\infty, x)s(x)$ if and only if $\mathbb{1}\overline{\otimes}x \geq \mu(\infty, \mathbb{1}\overline{\otimes}x)s(\mathbb{1}\overline{\otimes}x)$. The same is true for x_i. Therefore all the conditions in the proposition for x and x_i are equivalent to the corresponding conditions for $\mathbb{1}\overline{\otimes}x$ and $\mathbb{1}\overline{\otimes}x_i$.

We will show first that if $\mu(\infty, x_i) > 0$ then $s(x_i) = s(x)$, $i = 1, 2, \ldots, k+1$. Observe that $0 = n(x)xn(x) = \frac{1}{k+1}\sum_{i=1}^{k+1} n(x)x_i n(x)$, where $n(x)x_i n(x) \geq 0$. Hence $n(x)x_i n(x) = 0$, and so $x_i n(x) = 0$. Consequently $n(x_i) \geq n(x)$, $i = 1, 2, \ldots, k+1$. Thus by Lemma 3.6,

(3.3)
$$\begin{aligned} x + \mu(\infty, x)n(x) &= \frac{1}{k+1}\sum_{i=1}^{k+1} x_i + \mu(\infty, x)n(x) = \frac{1}{k+1}\sum_{i=1}^{k+1} x_i \\ &+ \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(\infty, x_i)n(x) \leq \frac{1}{k+1}\sum_{i=1}^{k+1}(x_i + \mu(\infty, x_i)n(x_i)). \end{aligned}$$

Note that by Proposition 1.5 (1), $\mu(x+\mu(\infty, x)n(x)) = \mu(x)$ and $\mu(x_i+\mu(\infty, x_i)n(x_i)) = \mu(x_i)$, $i = 1, 2, \ldots, k+1$. Furthermore, since $\mu(x)$ is k-extreme, Lemma 3.6 implies that $\mu(x) = \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(x_i)$. Therefore in view of (3.3),

$$\begin{aligned} \mu(x) = \mu(x + \mu(\infty, x)n(x)) &\leq \mu\left(\frac{1}{k+1}\sum_{i=1}^{k+1}(x_i + \mu(\infty, x_i)n(x_i))\right) \\ &\prec \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(x_i + \mu(\infty, x_i)n(x_i)) = \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(x_i) = \mu(x), \end{aligned}$$

and so

$$\mu(x + \mu(\infty, x)n(x)) = \mu\left(\frac{1}{k+1}\sum_{i=1}^{k+1}(x_i + \mu(\infty, x_i)n(x_i))\right).$$

However, $x+\mu(\infty, x)n(x) \leq \frac{1}{k+1}\sum_{i=1}^{k+1}(x_i+\mu(\infty, x_i)n(x_i))$ and $x+\mu(\infty, x)n(x) \geq \mu(\infty, x)\mathbf{1}$. By Proposition 1.5 (5) and in view of (3.3),

$$\begin{aligned} x + \mu(\infty, x)n(x) &= \frac{1}{k+1}\sum_{i=1}^{k+1}(x_i + \mu(\infty, x_i)n(x_i)) \\ &= \frac{1}{k+1}\sum_{i=1}^{k+1}(x_i + \mu(\infty, x_i)n(x)). \end{aligned} \tag{3.4}$$

Hence if $\mu(\infty, x_i) > 0$ then $n(x) = n(x_i)$ and $s(x) = s(x_i)$.

We will show next that $x_i \geq \mu(\infty, x_i)s(x_i)$, $i = 1, 2, \ldots, k+1$.

Assume first that $s(x) = \mathbf{1}$. Hence $n(x) = e^x\{0\} = 0$. Then $x \geq \mu(\infty, x)\mathbf{1}$ and in view of Lemma 3.4, $n(x) = e^x[0, \mu(\infty, x)) = 0$. Moreover, as observed in Lemma 3.6, $\mu(x) = \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(x_i)$.

Let $0 < \lambda < \tau(\mathbf{1}) = \infty$. Choose by Lemma 1.2 a projection p_λ such that $\tau(p_\lambda) = \lambda$ and

$$e^x(\mu(\lambda, x), \infty)) \leq p_\lambda \leq e^x[\mu(\lambda, x), \infty)). \tag{3.5}$$

By Lemma 1.3, $\tau(xp_\lambda) = \int_0^{\tau(p_\lambda)} \mu(x)$. Therefore

$$\frac{1}{k+1}\sum_{i=1}^{k+1} \tau(x_i p_\lambda) = \tau(xp_\lambda) = \int_0^{\tau(p_\lambda)} \mu(x) = \frac{1}{k+1}\sum_{i=1}^{k+1} \int_0^{\tau(p_\lambda)} \mu(x_i).$$

Since $\mu(x_i p_\lambda) \prec \mu(x_i)\chi_{[0,\tau(p_\lambda))}$, and so $\tau(x_i p_\lambda) = \int_0^\infty \mu(x_i p_\lambda) \le \int_0^{\tau(p_\lambda)} \mu(x_i)$, it follows that

$$\tau(x_i p_\lambda) = \int_0^{\tau(p_\lambda)} \mu(x_i),$$

for $i = 1, 2, \dots, k+1$. Consequently, Lemma 1.4 implies that

$$e^{x_i}\left(\mu(\lambda, x_i), \infty)\right) \le p_\lambda \le e^{x_i}\left[\mu(\lambda, x_i), \infty)\right), \quad i = 1, 2, \dots, k+1. \tag{3.6}$$

Denote by $p = \vee_{\lambda>0} p_\lambda$. Since $\vee_{\lambda>0} e^x\left(\mu(\lambda, x), \infty\right) = e^x\left(\mu(\infty, x), \infty\right)$ and $\vee_{\lambda>0} e^x\left[\mu(\lambda, x), \infty\right) \le e^x\left[\mu(\infty, x), \infty\right)$, relation (3.5) implies that

$$e^x\left(\mu(\infty, x), \infty\right) \le p \le e^x\left[\mu(\infty, x), \infty\right).$$

Similarly by (3.6),

$$e^{x_i}\left(\mu(\infty, x_i), \infty\right) \le p \le e^{x_i}\left[\mu(\infty, x_i), \infty\right), \quad i = 1, 2, \dots, k+1.$$

Let $j \in \{1, 2, \dots, k+1\}$ be fixed. Then

$$e^x\left(\mu(\infty, x), \infty\right) \le e^{x_j}\left[\mu(\infty, x_j), \infty\right), \tag{3.7}$$

and

$$e^{x_i}\left(\mu(\infty, x_i), \infty\right) \le e^{x_j}\left[\mu(\infty, x_j), \infty\right) \text{ for all } i = 1, 2, \dots, k+1. \tag{3.8}$$

Assume that $\mu(\infty, x_j) > 0$. We will show next that $e^{x_j}[0, \mu(\infty, x_j)) = 0$. Suppose to the contrary that $e^{x_j}[0, \mu(\infty, x_j)) \ne 0$. Choose $0 < \lambda_0 < \mu(\infty, x_j)$ such that $e^{x_j}[0, \lambda_0] \ne 0$. Since $\mathcal{M}$ is non-atomic and τ is semi-finite, there exists a non-zero projection $q \le e^{x_j}[0, \lambda_0]$ and $q \ne e^{x_j}[0, \lambda_0]$, with $\tau(q) < \infty$. Note first that by (3.7) and the fact that $e^x[0, \mu(\infty, x)) = 0$, $q \le e^{x_j}[0, \mu(\infty, x_j)) \le e^x[0, \mu(\infty, x)] = e^x\{\mu(\infty, x)\}$. Hence by spectral representation of x

$$xq = xe^x\{\mu(\infty, x)\}q = \mu(\infty, x)e^x\{\mu(\infty, x)\}q = \mu(\infty, x)q.$$

Consequently,

$$\begin{aligned}\sum_{i=1}^{k+1} \tau(x_i q) &= \tau\left(\sum_{i=1}^{k+1} x_i q\right) = \tau((k+1)xq) = (k+1)\tau(\mu(\infty, x)q) \\ &= \sum_{i=1}^{k+1} \mu(\infty, x_i)\tau(q).\end{aligned} \tag{3.9}$$

By (3.8), $q \le e^{x_j}[0, \mu(\infty, x_j)) \le e^{x_i}[0, \mu(\infty, x_i)]$ for all $i = 1, 2, \dots, k+1$. Since $0 \le x_i e^{x_i}[0, \mu(\infty, x_i)] \le \mu(\infty, x_i)e^{x_i}[0, \mu(\infty, x_i)]$ we have

$$\begin{aligned}\tau(x_i q) &= \tau(x_i e^{x_i}[0, \mu(\infty, x_i)]q) = \tau(q x_i e^{x_i}[0, \mu(\infty, x_i)]q) \\ &\le \tau(q\mu(\infty, x_i)e^{x_i}[0, \mu(\infty, x_i)]q) = \tau(\mu(\infty, x_i)e^{x_i}[0, \mu(\infty, x_i)]q) \\ &= \tau(\mu(\infty, x_i)q) = \mu(\infty, x_i)\tau(q),\end{aligned}$$

for all $i = 1, 2, \dots, k+1$. Therefore (3.9) implies that

$$\tau(x_i q) = \mu(\infty, x_i)\tau(q), \text{ for all } i = 1, 2, \dots, k+1.$$

In particular we get that $\tau(x_j q) = \mu(\infty, x_j)\tau(q)$. In view of $0 < \tau(q) < \infty$ we must have

$$\begin{aligned}\tau(x_j q) &= \tau(q x_j e^{x_j}[0,\lambda_0]q) \le \tau(q\lambda_0 e^{x_j}[0,\lambda_0]q) = \tau(\lambda_0 q) = \lambda_0\tau(q)\\ &< \mu(\infty, x_j)\tau(q),\end{aligned}$$

which is impossible. Consequently $e^{x_j}[0, \mu(\infty, x_j)) = 0$. By the first part of the proof, $n(x) = n(x_j) = e^{x_j}\{0\} = 0$, and by Lemma 3.4, $x_j \ge \mu(\infty, x_j)\mathbf{1}$. Since the same follows instantly for those x_i's for which $\mu(\infty, x_i) = 0$, we have that $x_i \ge \mu(\infty, x_i)\mathbf{1}$ for all $i = 1, 2, \dots, k+1$.

Consider now the general case, when $s(x)$ is not necessarily an identity. Recall that by Proposition 1.5 (1), $\mu(x+\mu(\infty,x)n(x)) = \mu(x)$ and $\mu(x_i+\mu(\infty,x_i)n(x_i) = \mu(x_i)$, $i = 1, 2, \dots, k+1$. We also have by (3.4) that $x+\mu(\infty,x)n(x) = \frac{1}{k+1}\sum_{i=1}^{k+1}(x_i+\mu(\infty,x_i)n(x_i))$. Clearly $s(x+\mu(\infty,x)n(x)) = \mathbf{1}$ and $x+\mu(\infty,x)n(x) \ge \mu(\infty,x)\mathbf{1}$.

We will repeat the above argument to the operators $x+\mu(\infty,x)n(x)$, $x_i + \mu(\infty,x_i)n(x_i)$ instead of x, x_i, respectively. Consequently, it follows that $x_i + \mu(\infty,x_i)n(x_i) \ge \mu(\infty,x_i)\mathbf{1}$ and $x_i \ge \mu(\infty,x_i)s(x_i)$ for all $i = 1, 2, \dots, k+1$. □

Lemma 3.8. *Suppose $\mu(x)$ is k-extreme, $x = \frac{1}{k+1}\sum_{i=1}^{k+1} b_i \le \frac{1}{k+1}\sum_{i=1}^{k+1} a_i$, $a_i, b_i \in B_{E(\mathcal{M},\tau)}$ and $a_i \prec b_i$, $i = 1, 2, \dots, k+1$. Then $\mu(a_i) = \mu(b_i)$ for all $i = 1, 2, \dots, k+1$.*

If in addition $\mu(\infty, x) = 0$ then $a_i = b_i$, for all $i = 1, 2, \dots, k+1$.

Proof. By Lemma 3.6 and in view of $a_i \prec b_i$, $\mu(x) = \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(b_i) \le \mu(\frac{1}{k+1}\sum_{i=1}^{k+1}a_i) \prec \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(a_i) \prec \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(b_i) = \mu(x)$. Hence $\mu(x) = \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(a_i) = \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(b_i)$, and so for all $t > 0$,

$$\sum_{i=1}^{k+1}\int_0^t (\mu(b_i) - \mu(a_i)) = 0.$$

Since for all $i = 1, 2, \dots, k+1$, $\mu(a_i) \prec \mu(b_i)$ we have that $\int_0^t (\mu(b_i) - \mu(a_i)) \ge 0$, $t > 0$. Therefore $\mu(a_i) = \mu(b_i)$, $i = 1, 2, \dots, k+1$.

Suppose now that $\mu(\infty, x) = 0$. Then clearly $\mu(\infty, a_i) = \mu(\infty, b_i) = 0$ for all $i = 1, 2, \dots, k+1$. Note that

$$x = \frac{1}{k+1}\sum_{i=1}^{k+1} b_i \le \frac{1}{k+1}\sum_{i=1}^{k+1}\frac{a_i+b_i}{2},$$

where $\frac{a_i+b_i}{2} \prec b_i$. By the previous argument, $\mu(b_i) = \mu\left(\frac{a_i+b_i}{2}\right)$. Therefore $\mu(a_i) = \mu(b_i) = \mu\left(\frac{a_i+b_i}{2}\right)$, and Proposition 1.5 (3) implies that $a_i = b_i$, $i = 1, 2, \dots, k+1$. □

Lemma 3.9. *Suppose that $\mu(x)$ is k-extreme, $|x| \ge \mu(\infty, x)s(x)$, and $x = \frac{1}{k+1}\sum_{i=1}^{k+1} a_i = \frac{1}{k+1}\sum_{i=1}^{k+1} b_i$, $a_i, b_i \in B_{E(\mathcal{M},\tau)}$. If $a_i \prec b_i$ and $a_i, b_i \ge 0$ for all $i = 1, 2, \dots, k+1$ then $a_i = b_i$, $i = 1, 2, \dots, k+1$.*

Proof. Note that if $a_i \ge 0$ for all $i = 1, 2, \dots, k+1$, then $x = |x|$. Since $|x| = \frac{1}{k+1}\sum_{i=1}^{k+1} a_i = \frac{1}{k+1}\sum_{i=1}^{k+1} b_i = \frac{1}{k+1}\sum_{i=1}^{k+1}\frac{a_i+b_i}{2}$, by Lemma 3.8,

$$\mu(a_i) = \mu(b_i) = \mu\left(\frac{a_i+b_i}{2}\right).$$

Denote by $C_i = \mu(\infty, a_i) = \mu(\infty, b_i) = \mu(\infty, \frac{a_i+b_i}{2})$. In case of $\mu(\infty, x) > 0$, Proposition 3.7 guarantees that if $C_i > 0$ then $a_i \geq C_i s(x)$, $b_i \geq C_i s(x)$ and $\frac{a_i+b_i}{2} \geq C_i s(x)$, and also $s(a_i) = s(b_i) = s(\frac{a_i+b_i}{2}) = s(x)$. Hence in view of Proposition 1.5 (2),

$$\mu(a_i - C_i s(x)) = \mu(a_i) - C_i = \mu(b_i) - C_i = \mu(b_i - C_i s(x)) = \mu\left(\frac{a_i+b_i}{2}\right)$$
$$-C_i = \mu\left(\frac{a_i+b_i}{2} - C_i s(x)\right) = \mu\left(\frac{a_i - C_i s(x) + b_i - C_i s(x)}{2}\right).$$

Clearly if $C_i = 0$ the above equalities hold. We have now that $a_i - C_i s(x), b_i - C_i s(x) \in S_0(\mathcal{M}, \tau)$, and so by Proposition 1.5 (3) it follows that $a_i = b_i$, $i = 1, 2, \ldots, k+1$. □

Lemma 3.10. *Suppose that $\mu(x)$ is k-extreme, $|x| = \frac{1}{k+1}\sum_{i=1}^{k+1} x_i$, $x_i \in B_{E(\mathcal{M},\tau)}$ and either $\mu(\infty, x) = 0$ or $|x| \geq \mu(\infty, x)s(x)$. Then $n(x)x_i = x_i n(x) = 0$ if and only if $x_i \geq 0$, $i = 1, 2, \ldots, k+1$.*

Proof. Assume first that for all $j = 1, 2, \ldots, k+1$, $n(x)x_j = x_j n(x) = 0$. Denote by $\mathrm{Re}\,(x_j) = \dfrac{x_j + x_j^*}{2}$, $j = 1, 2, \ldots, k+1$. Since $\mathrm{Re}\,(x_j) \leq |\mathrm{Re}\,(x_j)|$ we have that

$$|x| = \frac{1}{k+1}\sum_{j=1}^{k+1} x_j = \frac{1}{k+1}\sum_{j=1}^{k+1} \mathrm{Re}\,(x_j) \leq \frac{1}{k+1}\sum_{j=1}^{k+1} |\mathrm{Re}\,(x_j)|. \tag{3.10}$$

By Lemma 3.6, $\mu(x) \leq \mu(\frac{1}{k+1}\sum_{j=1}^{k+1} |\mathrm{Re}\,(x_j)|) \prec \frac{1}{k+1}\sum_{j=1}^{k+1} \mu(\mathrm{Re}\,(x_j)) = \mu(x)$, and so

$$\mu(x) = \mu\left(\frac{1}{k+1}\sum_{j=1}^{k+1} |\mathrm{Re}\,(x_j)|\right).$$

By assumption $x_j n(x) = n(x)x_j = 0$, and so $\mathrm{Re}\,(x_j)n(x) = n(x)\mathrm{Re}\,(x_j) = 0$. Thus $n(x)\left(\frac{1}{k+1}\sum_{j=1}^{k+1} \mathrm{Re}\,(x_j)\right) = \left(\frac{1}{k+1}\sum_{j=1}^{k+1} \mathrm{Re}\,(x_j)\right)n(x) = 0$. Denote by $C = \mu(\infty, x) = \mu\left(\infty, \frac{1}{k+1}\sum_{j=1}^{k+1} |\mathrm{Re}\,(x_j)|\right)$. By Corollary 1.6,

$$\mu(|x| + Cn(x)) = \mu(x) = \mu\left(\frac{1}{k+1}\sum_{j=1}^{k+1} |\mathrm{Re}\,(x_j)|\right)$$
$$= \mu\left(\frac{1}{k+1}\sum_{j=1}^{k+1} |\mathrm{Re}\,(x_j)| + Cn(x)\right).$$

Since $C\mathbf{1} \leq |x| + Cn(x) \leq \frac{1}{k+1}\sum_{j=1}^{k+1} |\mathrm{Re}\,(x_j)| + Cn(x)$, Proposition 1.5 (5) implies that $|x| = \frac{1}{k+1}\sum_{j=1}^{k+1} |\mathrm{Re}\,(x_j)| = \frac{1}{k+1}\sum_{j=1}^{k+1} \mathrm{Re}\,(x_j)$. Since for all $j = 1, 2, \ldots, k+1$, $\mathrm{Re}\,(x_j) \leq |\mathrm{Re}\,(x_j)|$ we get the equality $\mathrm{Re}\,(x_j) = |\mathrm{Re}\,(x_j)|$.

We will show next that $\mathrm{Im}\,(x_j) = 0$ and therefore $x_j = \mathrm{Re}\,(x_j)$. Note that $\mathrm{Re}\,(x_j) \prec x_j$, $j = 1, 2, \ldots, k+1$. Thus by Lemma 3.8 and (3.10) we have $\mu(x_j) = \mu(\mathrm{Re}\,(x_j))$. Let $C_j = \mu(\infty, x_j) = \mu(\infty, \mathrm{Re}\,(x_j))$, $j = 1, 2, \ldots, k+1$. Then

$$\mu(x_j + C_j n(x)) = \mu(x_j) = \mu(\mathrm{Re}\,(x_j)) = \mu(\mathrm{Re}\,(x_j) + C_j n(x)),$$

and therefore

$$\mu(\mathrm{Re}\,(x_j) + C_j n(x) + i\mathrm{Im}\,(x_j)) = \mu(\mathrm{Re}\,(x_j) + C_j n(x)).$$

Observe that by (3.10), if $\mu(\infty, x) > 0$ and $C_j > 0$, Proposition 3.7 implies that $\mathrm{Re}\,(x_j) + C_j n(x) \geq C_j \mathbf{1}$. Clearly, the same is true for $C_j = 0$ and also for the case $\mu(\infty, x) = 0$ since $\mu(x) = \frac{1}{k+1}\sum_{j=1}^{k+1}\mu(x_j)$. Thus by Proposition 1.5 (6) for all $j = 1, 2, \ldots, k+1$, $\mathrm{Im}\,(x_j) = 0$ and $x_j = \mathrm{Re}\,(x_j) \geq 0$.

For the converse assume that $x_j \geq 0$, for all $j = 1, 2, \ldots, k+1$. Then $0 = n(x)\,|x|\,n(x) = \frac{1}{k+1}\sum_{j=1}^{k+1} n(x) x_j n(x)$. Since $n(x)x_j n(x) \geq 0$ it follows that $n(x)x_j n(x) = 0$. Consequently $x_j n(x) = 0$ and $n(x)x_j = (x_j n(x))^* = 0$, for all $j = 1, 2, \ldots, k+1$. □

Lemma 3.11. *Suppose that $\mu(x)$ is k-extreme, $|x| = \frac{1}{k+1}\sum_{i=1}^{k+1} x_i$, $x_i \in B_{E(\mathcal{M},\tau)}$, $i = 1, 2, \ldots, k+1$, and $\mu(\infty, x) = 0$. Then $x_i \geq 0$ for all $i = 1, 2, \ldots, k+1$.*

Proof. Consider the equations

$$|x| = \frac{1}{k+1}\sum_{i=1}^{k+1} x_i = \frac{1}{k+1}\sum_{i=1}^{k+1} x_i^* \quad i = 1, 2, \ldots, k+1.$$

By Lemma 3.8, $x_i = x_i^*$, and consequently

$$|x| = \frac{1}{k+1}\sum_{i=1}^{k+1} x_i \leq \frac{1}{k+1}\sum_{i=1}^{k+1} |x_i|\,.$$

By Lemma 3.6, $\mu(x) = \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(x_i)$, and so $\mu(x) = \mu(\frac{1}{k+1}\sum_{i=1}^{k+1}|x_i|)$. As an immediate consequence of Proposition 1.5 (5) we have that $|x| = \frac{1}{k+1}\sum_{i=1}^{k+1}|x_i| = \frac{1}{k+1}\sum_{i=1}^{k+1} x_i$. Since $x_i \leq |x_i|$ the equality $x_i = |x_i|$ follows. □

Lemma 3.12. *Let $\mu(x)$ be k-extreme and $|x| = \frac{1}{k+1}\sum_{i=1}^{k+1}|x_i|$, $x_i \in B_{E(\mathcal{M},\tau)}$, $i = 1, 2, \ldots, k+1$. If $|x| \geq \mu(\infty, x)\mathbf{1}$ then for all $i = 1, 2, \ldots, k+1$,*

$$|x_i|\, e^{|x|}\{\mu(\infty, x)\} = \mu(\infty, x_i) e^{|x|}\{\mu(\infty, x)\}.$$

Proof. Observe first that in view of Lemma 3.6, if $\mu(\infty, x) = 0$ then we must have $\mu(\infty, x_i) = 0$ for all $i = 1, 2, \ldots, k+1$. Then the hypothesis becomes $|x_i|\, n(x) = 0$, and it follows by Lemma 3.10.

Assume now that $\mu(\infty, x) > 0$ and therefore $\mathbf{1} = s(x) = e^{|x|}[\mu(\infty, x), \infty)$ (see Lemma 3.4). We will show first that for all $i = 1, 2, \ldots, k+1$ we have

$$e^{|x|}\{\mu(\infty, x)\} \wedge e^{|x_i|}(\mu(\infty, x_i), \infty) = 0.$$

Fix $j \in \{1, 2, \ldots, k+1\}$ and assume that for some $s > \mu(\infty, x_j)$ the projection

$$p_j = e^{|x|}\{\mu(\infty, x)\} \wedge e^{|x_j|}(s, \infty) \neq 0.$$

Clearly $\tau(p_j) \leq \tau(e^{|x_j|}(s, \infty)) < \infty$ and

$$|x|\, p_j = |x|\, e^{|x|}\{\mu(\infty, x)\} p_j = \mu(\infty, x) e^{|x|}\{\mu(\infty, x)\} p_j = \mu(\infty, x) p_j.$$

Moreover, by Proposition 3.7 we have that for all $i = 1, 2, \ldots, k+1$, $|x_i| \geq \mu(\infty, x_i) s(x_i)$. We also have that if $\mu(\infty, x_i) > 0$ then $s(x_i) = s(x) = e^{|x|}[\mu(\infty, x), \infty)$, and so $p_j \leq s(x_i)$, $i = 1, 2, \ldots, k+1$. Hence it follows that

$$p_j\, |x_i|\, p_j \geq \mu(\infty, x_i) p_j s(x_i) p_j = \mu(\infty, x_i) p_j,$$

and $\tau(|x_i|\, p_j) = \tau(p_j\, |x_i|\, p_j) \geq \mu(\infty, x_i)\tau(p_j)$. Furthermore,

$$p_j\, |x_j|\, p_j = p_j\, |x_j|\, e^{|x_j|}(s,\infty)p_j \geq sp_j e^{|x_j|}(s,\infty)p_j = sp_j,$$

and consequently $\tau(|x_j|\, p_j) \geq s\tau(p_j) > \mu(\infty, x_j)\tau(p_j)$.

Recall that if $\mu(x)$ is k-extreme and $|x| = \frac{1}{k+1}\sum_{i=1}^{k+1}|x_i|$ then by Lemma 3.6, $\mu(x) = \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(x_i)$. Therefore

$$\mu(\infty,x)\tau(p_j) = \tau(|x|\, p_j) = \frac{1}{k+1}\sum_{i=1}^{k+1}\tau(|x_i|\, p_j) = \frac{1}{k+1}\sum_{i=1,\, i\neq j}^{k+1}\tau(|x_i|\, p_j)$$

$$+\frac{1}{k+1}\tau(|x_j|\, p_j) > \frac{1}{k+1}\sum_{i=1,\, i\neq j}^{k+1}\mu(\infty,x_i)\tau(p_j) + \frac{1}{k+1}\mu(\infty,x_j)\tau(p_j)$$

$$= \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(\infty,x_i)\tau(p_j) = \mu(\infty,x)\tau(p_j),$$

which contradicts the assumption that $p_j \neq 0$. Consequently $e^{|x|}\{\mu(\infty,x)\} \wedge e^{|x_j|}(s,\infty) = 0$ for all $s > \mu(\infty, x_j)$ and

$$e^{|x|}\{\mu(\infty,x)\} \wedge e^{|x_j|}(\mu(\infty,x_j),\infty) = 0.$$

If $\mu(\infty,x_j) = 0$, then clearly $e^{|x_j|}(\mu(\infty,x_j),\infty)^{\perp} = e^{|x_j|}\{\mu(\infty,x_j)\}$. On the other hand, if $\mu(\infty,x_j) > 0$, then by Proposition 3.7 and Lemma 3.4, $e^{|x_j|}\{0\} = n(x_j) = n(x) = 0$ and $e^{|x_j|}(0,\mu(\infty,x_j)) = 0$. Hence we also have $e^{|x_j|}(\mu(\infty,x_j),\infty)^{\perp} = e^{|x_j|}\{\mu(\infty,x_j)\}$. Thus $e^{|x|}\{\mu(\infty,x)\} = e^{|x|}\{\mu(\infty,x)\}\wedge(e^{|x_j|}(\mu(\infty,x_j),\infty)\vee e^{|x_j|}\{\mu(\infty,x_j)\}) = e^{|x|}\{\mu(\infty,x)\} \wedge e^{|x_j|}\{\mu(\infty,x_j)\} \leq e^{|x_j|}\{\mu(\infty,x_j)\}$. Therefore

$$\begin{aligned}
x_j e^{|x|}\{\mu(\infty,x)\} &= x_j e^{|x_j|}\{\mu(\infty,x_j)\}e^{|x|}\{\mu(\infty,x)\}\\
&= \mu(\infty,x_j)e^{|x_j|}\{\mu(\infty,x_j)\}e^{|x|}\{\mu(\infty,x)\}\\
&= \mu(\infty,x_j)e^{|x|}\{\mu(\infty,x)\},
\end{aligned}$$

and since j was arbitrary the claim follows. □

Theorem 3.13. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, σ- finite trace τ and E be a strongly symmetric function space. An element $x \in S_{E(\mathcal{M},\tau)}$ is a k-extreme point of $B_{E(\mathcal{M},\tau)}$ whenever $\mu(x)$ is a k-extreme point of B_E and one of the following conditions holds:*

(i) *$\mu(\infty,x) = 0$,*
(ii) *$n(x)\mathcal{M}n(x^*) = 0$ and $|x| \geq \mu(\infty,x)s(x)$.*

Proof. Assume first that $\mathcal{M}$ is non-atomic. Suppose that

$$x = \frac{1}{k+1}\sum_{i=1}^{k+1}x_i, \text{ where } x_i \in B_{E(\mathcal{M},\tau)}, \quad i = 1,2,\ldots,k+1. \tag{3.11}$$

Let $x = u\,|x|$ be the polar decomposition of x. Then

$$|x| = \frac{1}{k+1}\sum_{i=1}^{k+1}u^*x_i = \frac{1}{k+1}\sum_{i=1}^{k+1}u^*x_i s(x), \tag{3.12}$$

and

$$x = \frac{1}{k+1}\sum_{i=1}^{k+1} x_i = \frac{1}{k+1}\sum_{i=1}^{k+1} s(x^*)x_i. \tag{3.13}$$

Consider first the case when $|x| \geq \mu(\infty, x)\mathbf{1}$. Note that if $\mu(\infty, x) > 0$ then $s(x) = \mathbf{1}$ and $uu^* = u^*u = \mathbf{1}$.

If $\mu(\infty, x) = 0$ it follows by Lemma 3.11 and (3.12) that $u^*x_i \geq 0$ for all $i = 1, 2, \ldots, k+1$. Moreover, (3.13) combined with Lemma 3.8 implies that $s(x^*)x_i = x_i$.

If $\mu(\infty, x) > 0$ and consequently $n(x) = 0$, by Lemma 3.10 we also have that $u^*x_i \geq 0$. Clearly since $s(x^*) = \mathbf{1}$ we also have $x_i = s(x^*)x_i$ for all $i = 1, 2, \ldots, k+1$.

Furthermore in either case, $\mu(x_i) = \mu(uu^*x_i) \leq \mu(u^*x_i) \leq \mu(x_i)$, and so $\mu(x_i) = \mu(u^*x_i)$, $i = 1, 2, \ldots, k+1$. Therefore we always have

$$u^*x_i \geq 0, \quad s(x^*)x_i = x_i, \quad \text{and} \quad \mu(u^*x_i) = \mu(x_i), \quad i = 1, 2, \ldots, k+1. \tag{3.14}$$

Note that by Lemma 3.6, $\mu(x) = \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(x_i)$. Since $\mu(x)$ is k-extreme, there exist constants $C_1, C_2, \ldots, C_{k+1}$ not all vanishing such that

$$\sum_{i=1}^{k+1} C_i\mu(x_i) = 0. \tag{3.15}$$

Consider now the operator $|x| - \mu(\infty, x)\mathbf{1} \in S_0^+(\mathcal{M}, \tau)$ and denote by

$$p = s(|x| - \mu(\infty, x)\mathbf{1}) = e^{|x|}(\mu(\infty, x), \infty).$$

Lets define a projection q in the following way.

(1) $q = \mathbf{1}$ if $\tau(s(x)) < \infty$,
(2) $q = s(x)$ if $\tau(s(x)) = \infty$ and $\tau(p) < \infty$,
(3) $q = p$ if $\tau(s(x)) = \infty$ and $\tau(p) = \infty$.

Then by Corollary 1.10 there are a non-atomic, commutative von Neumann algebra $\mathcal{N} \subset q\mathcal{M}q$ and a $*$-isomorphism V from $S([0, \tau(\mathbf{1})), m)$ into $S(\mathcal{N}, \tau)$ such that

$$V\mu(x) = |x|\, q \quad \text{and} \quad \mu(V(f)) = \mu(f), \text{ for all } f \in S([0, \tau(\mathbf{1})), m).$$

By (3.12),

$$|x|\, q = \frac{1}{k+1}\sum_{i=1}^{k+1} qu^*x_iq,$$

and since $\mu(|x|\, q) = \mu(x)$ is k-extreme, Lemma 3.6 guarantees that

$$\frac{1}{k+1}\sum_{i=1}^{k+1}\mu(x_i) = \mu(x) = \mu(|x|\, q) = \frac{1}{k+1}\sum_{i=1}^{k+1}\mu(qu^*x_iq).$$

Clearly $\mu(qu^*x_iq) \leq \mu(x_i)$ and so $\mu(qu^*x_iq) = \mu(x_i)$, for all $i = 1, 2, \ldots, k+1$. Moreover applying V to the above equation we get the following

$$\frac{1}{k+1}\sum_{i=1}^{k+1} qu^*x_iq = |x|\, q = V\mu(x) = \frac{1}{k+1}\sum_{i=1}^{k+1} V\mu(qu^*x_iq).$$

We have $\mu(V\mu(qu^*x_iq)) = \mu(qu^*x_iq) = \mu(x_i)$, and $s(|x|\,q) = q$. By Lemma 3.9, $V\mu(qu^*x_iq) = qu^*x_iq$, $i = 1, 2, \ldots, k+1$. Applying now V to (3.15),

$$q\left(\sum_{i=1}^{k+1} C_i u^* x_i\right) q = \sum_{i=1}^{k+1} C_i q u^* x_i q = \sum_{i=1}^{k+1} C_i V\mu(x_i) = 0.$$

Hence $\sum_{i=1}^{k+1} C_i u^* x_i q = \left(\sum_{i=1}^{k+1} C_i u^* x_i\right) q = 0$ and in view of (3.14), $s(x^*)x_i = x_i$, and consequently

$$\sum_{i=1}^{k+1} C_i x_i q = 0. \tag{3.16}$$

Case 1. Let $\tau(s(x)) < \infty$ and $q = \mathbf{1}$. Clearly by (3.16),

$$\sum_{i=1}^{k+1} C_i x_i = 0.$$

Case 2. Assume that $\tau(s(x)) = \infty$, $\tau(p) < \infty$ and $q = s(x)$. Note that if $\mu(\infty, x) = 0$ then $p = s(x)$. Therefore the case is only possible when $\mu(\infty, x) > 0$ and then $s(x) = \mathbf{1}$. Therefore $q = \mathbf{1}$ and again by (3.16),

$$\sum_{i=1}^{k+1} C_i x_i = 0.$$

Case 3. Assume that $\tau(s(x)) = \infty$, $\tau(p) = \infty$ and $q = p$. If $\mu(\infty, x) = 0$, $q = e^{|x|}(0, \infty)$ and $q^\perp = e^{|x|}\{0\} = e^{|x|}\{\mu(\infty, x)\}$. If $\mu(\infty, x) > 0$ and so $n(x) = e^{|x|}\{0\} = 0$, we also have that $q^\perp = e^{|x|}\{\mu(\infty, x)\}$.

Therefore by Lemma 3.12 in view of (3.12) and (3.14) we get $u^*x_iq^\perp = \mu(\infty, x_i)q^\perp$. By (3.15),

$$\sum_{i=1}^{k+1} C_i u^* x_i q^\perp = \left(\sum_{i=1}^{k+1} C_i \mu(\infty, x_i)\right) q^\perp = 0.$$

Since by (3.14), $s(x^*)x_i = uu^*x_i = x_i$, the above equation becomes

$$\sum_{i=1}^{k+1} C_i x_i q^\perp = 0.$$

This combined with (3.16) implies that

$$\sum_{i=1}^{k+1} C_i x_i = 0.$$

Consequently in either case $x_1, x_2, \ldots, x_{k+1}$ are linearly dependent, and x is k-extreme.

We have shown that if $\mu(x)$ is a k-extreme point of B_E and $|x| \geq \mu(\infty, x)\mathbf{1}$, then x is a k-extreme point of $B_{E(\mathcal{M},\tau)}$. In particular if $\mu(\infty, x) = 0$, the claim follows.

Assume now that $\mu(\infty, x) > 0$ and (ii) holds. By Lemma 3.6 and equations (3.11) and (3.12),

$$\mu(x) = \frac{1}{k+1}\sum_{i=1}^{k+1} \mu(x_i) = \frac{1}{k+1}\sum_{i=1}^{k+1} \mu(u^* x_i s(x)).$$

Since $\mu(u^*x_is(x)) \leq \mu(x_i)$, the equality $\mu(u^*x_is(x)) = \mu(x_i)$ holds for all $i = 1, 2, \ldots, k+1$. Moreover by (3.11),

$$\begin{aligned}|x| + \mu(\infty, x)n(x) &= \frac{1}{k+1}\sum_{i=1}^{k+1} u^*x_is(x) + \mu(\infty, x)n(x)\\ &= \frac{1}{k+1}\sum_{i=1}^{k+1} \left(u^*x_is(x) + \mu(\infty, x_i)n(x)\right).\end{aligned}$$

Clearly $n(x)u^*x_is(x) = u^*x_is(x)n(x) = 0$, and $\mu(\infty, x_i) = \mu(\infty, u^*x_is(x))$. Hence by Corollary 1.6, $\mu\left(u^*x_is(x) + \mu(\infty, x_i)n(x)\right) = \mu(u^*x_is(x)) = \mu(x_i)$, and so $u^*x_is(x)+ \mu(\infty, x_i)n(x) \in B_{E(\mathcal{M},\tau)}$, $i = 1, 2, \ldots, k+1$. Since $\mu(|x| + \mu(\infty, x)n(x)) = \mu(x)$ is k-extreme, where $|x| + \mu(\infty, x)n(x) \geq \mu(\infty, x)\mathbf{1}$, the previous case implies that $|x| + \mu(\infty, x)n(x)$ is k-extreme. Furthermore it follows from the first part of the proof that for the constants $C_1, C_2, \ldots, C_{k+1}$, not all equal to zero and such that

$$\sum_{i=1}^{k+1} C_i\mu\left(u^*x_is(x) + \mu(\infty, x_i)n(x)\right) = \sum_{i=1}^{k+1} C_i\mu(x_i) = 0,$$

we have the corresponding equality for operators with the same constants C_i's, that is

$$\sum_{i=1}^{k+1} C_i\left(u^*x_is(x) + \mu(\infty, x_i)n(x)\right) = 0.$$

Since clearly $\sum_{i=1}^{k+1} C_i\mu(\infty, x_i) = 0$, we have in fact that

$$\sum_{i=1}^{k+1} C_iu^*x_is(x) = 0.$$

Recall that $s(x^*) = uu^*$, $u^* = s(x)u^* = u^*uu^* = u^*s(x^*)$. Multiplying the above sum by u from the left and u^* from the right we get

$$\sum_{i=1}^{k+1} C_is(x^*)x_iu^*s(x^*) = s(x^*)\left(\sum_{i=1}^{k+1} C_ix_iu^*\right)s(x^*) = 0.$$

Thus $\sum_{i=1}^{k+1} C_ix_iu^*s(x^*) = \sum_{i=1}^{k+1} C_ix_iu^* = 0$, and

$$\sum_{i=1}^{k+1} C_ix_is(x) = 0. \tag{3.17}$$

Observe that $x^* = \frac{1}{k+1}\sum_{i=1}^{k+1} x_i^*$ and $|x^*| = u^*x^*$. Repeating the same argument as above for x^* and x_i^*'s instead of x and x_i's, respectively, and using the complex conjugate of the equality (3.15), $\sum_{i=1}^{k+1}\overline{C_i}\mu(x_i) = 0$ we get $\sum_{i=1}^{k+1}\overline{C_i}x_i^*s(x^*) = 0$. Hence

$$\sum_{i=1}^{k+1} C_is(x^*)x_i = 0. \tag{3.18}$$

Consequently combining (3.17) and (3.18),

$$\sum_{i=1}^{k+1} C_ix_i = \sum_{i=1}^{k+1} C_in(x^*)x_in(x).$$

Since by [9, Lemma 3.3] the assumption $n(x)\mathcal{M}n(x^*) = 0$ implies that $n(x)S(\mathcal{M},\tau)\, n(x^*) = 0$, so $n(x^*)x_i n(x) = 0$ for all $i = 1, 2, \ldots, k+1$. Therefore

$$\sum_{i=1}^{k+1} C_i x_i = 0,$$

and x is a k-extreme point of $B_{E(\mathcal{M},\tau)}$.

Let us suppose now that $\mathcal{M}$ is not non-atomic, $\mu(x)$ is k-extreme, and (i) and (ii) hold. Consider a non-atomic von Neumann algebra $\mathcal{A}$ with the trace κ, discussed in Remark 1.11. Then $\mathbb{1}\overline{\otimes}x \in S(\mathcal{A},\kappa)$, $\tilde{\mu}(\mathbb{1}\overline{\otimes}x) = \mu(x)$, and (i), (ii) are satisfied for the operator $\mathbb{1}\overline{\otimes}x$ (see the proof of Proposition 3.7). Hence by the first part of the proof, $\mathbb{1}\overline{\otimes}x$ is a k-extreme point of $B_{E(\mathcal{A},\kappa)}$. Since $\mathbb{1}\overline{\otimes}x = \tilde{\pi}(x)$, where $\tilde{\pi}$ is an isometry from $E(\mathcal{M},\tau)$ onto the subspace $E(\mathbb{C}\mathbb{1}\otimes\mathcal{M},\kappa)$ of $E(\mathcal{A},\kappa)$, it follows easily that x is a k-extreme point of $B_{E(\mathcal{M},\tau)}$. □

Combining now the results of Theorems 3.5 and 3.13, we give a complete characterization of k-extreme points in terms of their singular value functions, when $\mathcal{M}$ is a non atomic von Neumann algebra. For $k = 1$ we obtain the well-known theorem on extreme points proved in [6].

Theorem 3.14. *Let E be a strongly symmetric space on $[0,\tau(\mathbf{1}))$ and $\mathcal{M}$ be a non-atomic, semifinite von Neumann algebra with a faithful, normal, σ-finite trace τ. An operator x is a k-extreme point of $B_{E(\mathcal{M},\tau)}$ if and only if $\mu(x)$ is a k-extreme point of B_E and one of the following, not mutually exclusive, conditions holds:*

(i) *$\mu(\infty,x) = 0$;*
(ii) *$n(x)\mathcal{M}n(x^*) = 0$ and $|x| \geq \mu(\infty,x)s(x)$.*

Since in the commutative settings for any operator x, $n(x) = n(x^*)$, the conditions $n(x)\mathcal{M}n(x^*) = 0$ and $|x| \geq \mu(\infty,x)s(x)$ reduce to $|x| \geq \mu(\infty,x)\mathbf{1}$. Therefore by the above theorem applied to the commutative von Neumann algebra $\mathcal{M} = L_\infty[0,\tau(\mathbf{1}))$ the following holds.

Corollary 3.15. *Let E be a strongly symmetric function space. The following conditions are equivalent:*

(i) *f is a k-extreme point of B_E;*
(ii) *$\mu(f)$ is a k-extreme point of B_E and $|f| \geq \mu(\infty,f)$.*

The following simple observation will be useful in relating k-rotundity of E and $E(\mathcal{M},\tau)$.

Lemma 3.16. *If E is a k-rotund symmetric function space then $E = E_0$, that is $\mu(\infty,f) = 0$ for all $f \in E$.*

Proof. Suppose to the contrary that $E \neq E_0$, and so $\chi_{(0,\infty)} \in E$. Without loss of generality we can assume that $\|\chi_{(0,\infty)}\|_E = 1$. Let $f = \chi_{(k+1,\infty)}$. Then $\mu(f) = \chi_{(0,\infty)}$ and $f \in S_E$. For $i = 1, 2, \ldots, k$, define

$$u_i = -\frac{1}{k}\chi_{(0,1)} + \frac{1}{k}\chi_{(i,i+1)}.$$

Clearly, $\mu(f+u_i)=\chi_{(0,\infty)}$, and so $f+u_i\in S_E$. Moreover,

$$\left|f-\sum_{i=1}^k u_i\right| = \left|\chi_{(0,1)}-\frac{1}{k}\chi_{(1,k+1)}+\chi_{(k+1,\infty)}\right| = \chi_{(0,1)}+\frac{1}{k}\chi_{(1,k+1)} + \chi_{(k+1,\infty)} \le \chi_{(0,\infty)},$$

and also $f-\sum_{i=1}^k u_i\in B_E$. However $u_1,u_2,\dots,u_k$ are linearly independent, which in view of Proposition 2.2 implies that f cannot be k-extreme. □

Corollary 3.17. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra, with a faithful, normal, semi-finite trace τ. If a symmetric space E is k-rotund then $E(\mathcal{M},\tau)$ is k-rotund. If in addition $\mathcal{M}$ is non-atomic, then k-rotundity of $E(\mathcal{M},\tau)$ implies k-rotundity of E.*

Proof. If E is k-rotund, then by Lemma 3.16 we have that $E=E_0$. Let $x\in S_{E(\mathcal{M},\tau)}$ and $x=\frac{1}{k+1}\sum_{i=1}^{k+1}x_i$, $x_i\in B_{E(\mathcal{M},\tau)}$. Then $\mu(x)$ is k-extreme. Since $E=E_0$, $s(x)$ and $s(x_i)$, $i=1,2,\dots,k+1$ are σ-finite projections. Set $p=\vee_{i=1}^{k+1}s(x_i)\vee s(x)\vee s(x_i^*)\vee s(x^*)$. Then $pxp=ps(x^*)xs(x)p=s(x^*)xs(x)=x$ and $px_ip=x_i$, $i=1,2,\dots,k+1$. Hence x,x_i, $i=1,2,\dots,k+1$, belong to the subspace which is isometric to $E(\mathcal{M}_p,\tau_p)$, where τ_p is σ-finite. By Theorem 3.13 (i), we get that x is k-extreme in $E(\mathcal{M}_p,\tau_p)$ and $x_1,x_2,\dots,x_{k+1}$ are linearly dependent in $E(\mathcal{M}_p,\tau_p)$. Hence $x_1,x_2,\dots,x_{k+1}$ are linearly dependent in $E(\mathcal{M},\tau)$, and so x is k- extreme point of $B_{E(\mathcal{M},\tau)}$. Consequently $E(\mathcal{M},\tau)$ is k-rotund.

Suppose now that $\mathcal{M}$ is non-atomic and $E\,(\mathcal{M},\tau)$ is k-rotund. Then $E(\mathcal{M}_p,\tau_p)$ is k-rotund for any projection $p\in P(\mathcal{M})$. Let $p\in P(\mathcal{M})$ be a σ-finite projection with $\tau(p)=\tau(\mathbf{1})$ (see e.g. [8, Lemma 1.13]). By Proposition 1.8, E is isometrically embedded in $E(\mathcal{M}_p,\tau_p)$ and therefore E is also k-rotund. □

4. Orbits and Marcinkiewicz spaces

We finish with a characterization of k-extreme points in the orbits of functions. Letting $g\in L_1[0,\alpha)+L_\infty[0,\alpha)$, $\alpha\le\infty$, the *orbit* of g is the set $\Omega(g)=\{f\in L_1[0,\alpha)+L_\infty[0,\alpha): f\prec g\}$ [21]. Clearly the inequality $f\prec g$ is equivalent to

$$\|f\|_{M_G}:=\sup_{t>0}\frac{\int_0^t\mu(f)}{\int_0^t\mu(g)}\le 1.$$

Setting $G(t)=\int_0^t\mu(g)$, the *Marcinkiewicz* space M_G is the set of all $f\in L^0$ such that $\|f\|_{M_G}<\infty$ [17, 18]. The space M_G equipped with the norm $\|\cdot\|_{M_G}$ is a strongly symmetric function space. Therefore the orbit $\Omega(g)$ is the unit ball B_{M_G} in the space M_G.

Theorem 4.1. *Let $g\in L_1[0,\alpha)+L_\infty[0,\alpha)$. Then the following are equivalent.*

(i) *f is an extreme point of $\Omega(g)$.*
(ii) *f is a k-extreme point of $\Omega(g)$.*
(iii) *$\mu(f)$ is a k-extreme point of $\Omega(g)$ and $|f|\ge\mu(\infty,f)$.*
(iv) *$\mu(f)=\mu(g)$ and $|f|\ge\mu(\infty,f)$.*

Proof. Clearly (i) implies (ii), and (ii) and (iii) are equivalent by Corollary 3.15. The implication (iii) to (iv) follows by Corollary 2.7. We will show next that $\mu(g)$ is an extreme point of $\Omega(g)$. Consequently if (iv) holds, $\mu(f)=\mu(g)$ is an extreme point of $\Omega(f)$ and by Corollary 3.15, (i) follows.

Let $\mu(g) = \frac{1}{2}h_1 + \frac{1}{2}h_2$, where $h_1, h_2 \in \Omega(g)$. Then for all $s \in (0,\alpha)$ we have that

$$\begin{aligned}\int_0^s \mu(g) &= \frac{1}{2}\int_0^s h_1 + \frac{1}{2}\int_0^s h_2 \le \frac{1}{2}\int_0^s \mu(h_1) + \frac{1}{2}\int_0^s \mu(h_2)\\ &\le \frac{1}{2}\int_0^s \mu(g) + \frac{1}{2}\int_0^s \mu(g) = \int_0^s \mu(g).\end{aligned}$$

Hence $h_1 = \mu(h_1) = \mu(g) = \mu(h_2) = h_2$, and $\mu(g)$ is an extreme point of $\Omega(g)$. □

As an immediate consequence we get the following result.

Corollary 4.2. *Let M_G be the Marcinkiewicz space and k be any natural number. The function f is a k-extreme point of B_{M_G} if and only if $\mu(f) = \mu(g)$ and $|f| \ge \mu(\infty, f)$. Consequently f is a k-extreme point of B_{M_G} if and only if f is an extreme point of B_{M_G}.*

We conclude the paper with a description of k-extreme points for another important class of orbits $\Omega'(g)$, $0 \le g \in L_1[0,\alpha)$, $\alpha < \infty$ [20, 23].

Recall that, given $0 \le g \in L_1[0,\alpha)$, $\alpha < \infty$, the orbit $\Omega'(g)$ is defined as

$$\Omega'(g) = \{0 \le f \in L_1[0,\alpha) : \ f \prec g \quad \text{and} \quad \|f\|_1 = \|g\|_1\}.$$

Lemma 4.3. *Let $0 \le g \in L_1[0,\alpha)$, $\alpha < \infty$. Then f is a k-extreme point of $\Omega'(g)$ if and only if $f \ge 0$ and f is a k-extreme point of $\Omega(g)$.*

Proof. Suppose that $f \ge 0$ is a k-extreme point of $\Omega(g)$. Then by Theorem 4.1, $\mu(f) = \mu(g)$ and so $f \in \Omega'(g)$. Consequently, f is a k-extreme point of $\Omega'(g)$.

Assume now that f is a k-extreme point of $\Omega'(g)$. Let $f = \frac{1}{k+1}\sum_{i=1}^{k+1} f_i$ where $f_i \in \Omega(g)$. Since $f \ge 0$, $\int_0^\alpha f = \int_0^\alpha g$ and $f_i \prec g$, we have

$$\int_0^\alpha g = \int_0^\alpha f = \int_0^\alpha \mu(f) \le \frac{1}{k+1}\sum_{i=1}^{k+1}\int_0^\alpha \mu(f_i) \le \int_0^\alpha \mu(g) = \int_0^\alpha g.$$

Hence, $\int_0^\alpha \mu(f_i) = \int_0^\alpha g$, for $i = 1, 2, \ldots, k+1$.

Since $0 \le f = \frac{1}{k+1}\sum_{i=1}^{k+1} f_i \le \frac{1}{k+1}\sum_{i=1}^{k+1} |f_i|$ and

$$\begin{aligned}\int_0^\alpha g = \int_0^\alpha f &= \int_0^\alpha \left(\frac{1}{k+1}\sum_{i=1}^{k+1} f_i\right) \le \int_0^\alpha \left(\frac{1}{k+1}\sum_{i=1}^{k+1} |f_i|\right)\\ &= \frac{1}{k+1}\sum_{i=1}^{k+1}\|f_i\|_1 = \int_0^\alpha g,\end{aligned}$$

it follows that $f_i = |f_i|$ and $f_i \in \Omega'(g)$, $i = 1, 2, \ldots, k+1$. Therefore $f_1, f_2, \ldots, f_{k+1}$ are linearly dependent and f is a k-extreme point of $\Omega(g)$. □

The above lemma and Theorem 4.1 show that the sets of extreme and k-extreme points for $\Omega'(g)$, $0 \le g \in L_1[0,\alpha)$, $\alpha < \infty$, coincide. Consequently, the description of extreme points of $\Omega'(g)$ presented in [23] applies also for k-extreme points.

Proposition 4.4. *Let $0 \le g \in L_1[0,\alpha)$, $\alpha < \infty$. Then f is a k-extreme point of $\Omega'(g)$ if and only if $\mu(f)$ is a k-extreme point of $\Omega'(g)$. Moreover, the set of all k-extreme points of $\Omega'(g)$ is given by*

$$k\text{-}ext(\Omega'(g)) = \{0 \le f \in L_1[0,\alpha) : \ \mu(f) = \mu(g)\}.$$

References

1. J. Arazy, *On the geometry of the unit ball of unitary matrix spaces*, Integral Equations Operator Theory **4** (1981), no. 2, 151–171.
2. P. Bandyopadhyay, V. P. Fonf, B. L. Lin and M. Martín, *Structure of nested sequences of balls in Banach spaces*, Houston J. Math., **29** (2003), no. 1, 173–193.
3. C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics, vol. 129, Academic Press Inc., Boston, MA, 1988.
4. S. Chen, *Geometry of Orlicz spaces*, Dissertationes Math. (Rozprawy Mat.), **356** (1996).
5. V. I. Chilin, P. G. Dodds and F. A. Sukochev, *The Kadets-Klee property in symmetric spaces of measurable operators*, Israel J. Math. **97** (1997), 203–219.
6. V. I. Chilin, A. V. Krygin, and F. A. Sukochev, *Extreme points of convex fully symmetric sets of measurable operators*, Integral Equations Operator Theory **15** (1992), no. 2, 186–226.
7. V. I. Chilin, A. V. Krygin, and F. A. Sukochev, *Local uniform and uniform convexity of noncommutative symmetric spaces of measurable operators*, Math. Proc. Cambridge Philos. Soc. **111** (1992), no. 2, 355–368.
8. M. M. Czerwińska, *Geometric properties of symmetric spaces of measurable operators*, (Order No. 3476359, The University of Memphis), ProQuest Dissertations and Theses, **131**, 2011; http://search.proquest.com/docview/893659843.
9. M. M. Czerwińska and A. Kamińska *Complex rotundity properties and midpoint local uniform rotundity in symmetric spaces of measurable operators*, Studia Math. **201** (2010), no. 3, 253–285.
10. M. M. Czerwińska, A. Kamińska and D. Kubiak, *Smooth and strongly smooth points in symmetric spaces of measurable operators*, Positivity, **16** (2012), no. 1, 29–51.
11. P. G. Dodds, T. K. Dodds, and B. De Pagter, *Noncommutative Banach function spaces*, Math. Z. **201** (1989), no. 4, 583–597.
12. P. G. Dodds, T. K. Dodds, and B. De Pagter, *Noncommutative Köthe duality*, Trans. Amer. Math. Soc. **339** (1993), no. 2, 717–750.
13. P. G. Dodds, B. De Pagter and F. A. Sukochev, *Theory of Noncommutative Integration*, unpublished monograph; to appear.
14. T. Fack and H. Kosaki, *Generalized s-numbers of τ-measurable operators*, Pacific J. Math. **123** (1986), no. 2, 269–300.
15. R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras. Vol. I*, Graduate Studies in Mathematics, vol. 15, American Mathematical Society, Providence, RI, 1997.
16. N. J. Kalton and F. A. Sukochev, *Symmetric norms and spaces of operators*, J. Reine Angew. Math. **621** (2008), 81–121.
17. A. Kamińska, A. Parrish, *Note on extreme points in Marcinkiewicz function spaces*, Banach J. Math. Anal., **4** (2010), no. 1., 1–12.
18. S. G. Kreĭn, Yu. Ī. Petunīn, and E. M. Semënov, *Interpolation of Linear Operators*, Translations of Mathematical Monographs, vol. 54, American Mathematical Society, Providence, R.I., 1982.
19. B. De Pagter, *Non-commutative Banach function spaces*, Positivity, Trends Math., Birkhäuser, Basel, 2007, pp. 197–227.
20. Y. Raynaud, Q. Xu, *On subspaces of non-commutative L_p-spaces*. J. Funct. Anal. **203** (2003), no. 1, 149–196.
21. J. V. Ryff, *Extreme points of some convex subsets of $L^1(0,1)$*, Proc. Amer. Math. Soc., **18** (1967), 1026–1034.
22. F. A. Sukochev and V. I. Chilin, *The triangle inequality for operators that are measurable with respect to Hardy-Littlewood order*, Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk (1988), no. 4, 44–50.
23. F. Sukochev and D. Zanin, *Orbits in symmetric spaces*, J. Funct. Anal. **257** (2009), no. 1, 194–218.
24. W. F. Stinespring, *Integration theorems for gages and duality for unimodular groups*, Trans. Amer. Math. Soc. **90** (1959), 15–56.
25. M. Takesaki, *Theory of Operator Algebras. I*, Springer-Verlag, New York, 1979.
26. Liu Zheng and Zhuang Ya-Dong, *k-Rotund complex normed linear spaces*, J. Math. Anal. Appl. **146** (1990), 540–545.

27. Zhuang Ya-Dong, *On k-rotund complex normed linear spaces*, J. Math. Anal. Appl. **174** (1993), 218–230.

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